DANIELE ANGELLA, "NON-KÄHLER METRICS AND COHOMOLOGIES ON CPLX MFDS"
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ABSTRACT: provide tools for studying cohomologies of non-Kähler manifolds, to the aim to investigate connections between ~~exotic~~ geometric aspects (existence of special metrics) and algebraic structure of cohomology.

THM (WEIL). (X^{2n}, J, g, ω) cpt Kähler,
i.e., holonomy reduces to ~~U(n)~~
$$U(n) = SO(2n) \cap Sp(2n) \cap GL(n;\mathbb{C}).$$
Then:
$$H^\bullet_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X;\mathbb{C})$$
where $H^{p,q}(X;\mathbb{C}) \simeq H^{p,q}_{\bar\partial}(X) \simeq \check{H}^q(X;\Omega^p_X)$
$$= \bigoplus_{r\geq 0} L^r PH^{k-2r}$$

It follows, e.g., that odd Betti numbers are even.
In fact:

THM (DELIGNE-GRIFFITHS-MORGAN-SULLIVAN).
X cpt Kähler. Then it satisfies the $\partial\bar\partial$-Lemma.
Then the dga $(\wedge^\bullet X, d)$ is formal.

Recall:

- (X, J) cplx satisfies $\partial\bar\partial$-Lemma if the map
$$\text{(+)} \quad H^{\bullet\bullet}_{BC}(X) := \frac{\ker \partial \cap \ker \bar\partial}{\operatorname{im} \partial\bar\partial} \to H^{\bullet}_{dR}(X;\mathbb{C})$$
induced by identity is isomorphism.

(In a sense, $\partial\bar\partial$-Lemma yields quantitative properties of the cohomology, by connecting cplx aspects ($H^{\bullet\bullet}_{BC}(X)$) with topological aspects ($H^{\bullet}_{dR}(X;\mathbb{C}) \simeq \check{H}^{\bullet}(X;\underline{\mathbb{C}}_X)$) of cohomologies.)

- The structure $(\wedge^{\bullet}X, d)$ of differential $\mathbb{Z}$-graded algebra (dga) induces a structure of algebra on $H^{\bullet}_{dR}(X;\mathbb{C})$.
<u>But</u>: once you fix representatives of ~~H_{dR}~~ classes in dR-cohom, in general you have just A_∞-alg (ie, "algebra up to homotopy", higher ~~dimensions~~ products being representatives of Massey products).
You call X <u>formal</u> if, for a choice of representat, you get a genuine algebra. (When this happens for harmonic representatives wrt a metric, then X is called <u>geometrically-formal</u>.)

(In a sense, formality yields qualitative properties concerning the algebraic structure of cohomology.)

Summarizing:

eg. Kähler $\Longrightarrow$ - Hodge decomposition
- formality

~~GEOMETRY~~

METRIC PROPERTIES $\rightleftarrows$ COHOMOLOGICAL PROPERTIES

~~PRINCIPAL AIM~~

LONG-TERM AIM:

~~that~~ investigate how cohomological properties affect geometric properties.

(Many possible statement are still vague "conjectures.")

THM (GAUDUCHON). Let X be cpt cplx. In any conformal class of Hermitian metric, there is a $\underset{}{\overset{\text{(Gauduchon)}}{\text{standard}}}$ metric st $\partial\bar{\partial}\omega^{n-1}=0$.

THM (POPOVICI). Furthermore, if $\partial\bar{\partial}$-Lemma holds, then it is in fact strongly-Gauduchon

(i.e., the $(n-1,n-1)$-component of a closed form).

THM (d. Angella – SIMONE CALAMAI – CRISTIANO SPOTTI).

Let X be cpt cplx with $\mathrm{Kod}(X) \geq 0$.

Then, in any conformal class of Hermitian metrics, there is a metric with constant scalar curvature wrt Chern connection

QUESTION $\partial\bar{\partial}$-Lemma $\overset{?}{\Longrightarrow}$ $\exists$ balanced metric?

PROBEM relation between: sG $\longleftrightarrow$ balanced

holom-toned $\longleftrightarrow$ Kähler

SHORT-TERM AIM:

to better understand the complex and symplectic contribution to cohomological property for non-Kähler mfd.

WHY non-Kähler: ~~THURSTON~~

[THURSTON, PROC. AMS, 1976]: many examples

[DE BARTOLOMEIS-TOMASSINI, ANNALES INST. FOURIER, 2013]: better understandment of Kähler geom, too

[TSENG-YAU, COMM. MATH. PHYS., 2014]: string theory

HOW: frame cplx and sympl structures as generalized-cplx

ABSTRACT 2. We investigate cohom. properties of generalized-cplx structures

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First of all: what is $H^{pq}(X)$ in Hodge decomposition? I mean: how to relate cplx & top cohom? Note that there is no natural map between $H_{\bar\partial}(X)$ and $H_{dR}(X;\mathbb{C})$, in general. But, if you def

$$H^{\bullet\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\bar\partial}{\operatorname{im}\partial\bar\partial} \longrightarrow H^{\bullet}_{dR}(X;\mathbb{C})$$

then this makes sense.

So, you can consider

$$H^{pq}(X) := \operatorname{im}(H^{pq}_{BC} \to H_{dR}(X;\mathbb{C}))$$

$$\subseteq H^{p+q}_{dR}(X;\mathbb{C})$$

CPLX GEOM.	GENERALIZED-COMPLEX GEOM. [HITCHIN], [GUALTIERI], [CAVALCANTI], ...	SYMPLECTIC GEOM.
DEF. $J : TX \xrightarrow{\simeq} TX$ alg: $J^2 = -\mathrm{id}$ and: $\mathrm{Nij}_J = 0$	DEF. $\mathcal{J} : TX \oplus T^*X \to TX \oplus T^*X$ alg: $\mathcal{J}^2 = -\mathrm{id}$ and: $\mathrm{Nij}_{\mathcal{J}} = 0$ cptb: $\mathcal{J}$ orthog wrt $\langle X+\xi \mid Y+\eta \rangle := \frac{1}{2}(\xi(Y)+\eta(X))$	DEF. $\omega : TX \xrightarrow{\simeq} T^*X$ alg: ω non-deg 2-form and: $d\omega = 0$
$\mathcal{J}_J = \left(\begin{array}{c\|c} -J & \\ \hline & J^* \end{array}\right)$		$\mathcal{J}_\omega = \left(\begin{array}{c\|c} & -\omega^{-1} \\ \hline \omega & \end{array}\right)$
$(\wedge^{\bullet,\bullet}X, \partial, \bar\partial)$ double cplx (bidiff $\mathbb{Z}^2$-grad alg) $U^k = \bigoplus_{p+q=k} \wedge^{p,q}X$ $\partial_{\mathcal{J}} = \partial$ $\bar\partial_{\mathcal{J}} = \bar\partial$	$(\wedge X = \bigoplus_{k\in\mathbb{Z}} U^k,\ d = \partial_{\mathcal{J}} + \bar\partial_{\mathcal{J}})$ bi-diff $\mathbb{Z}$-graded vect-sp where: $U^n := \langle\rho\rangle$ where ρ is annihilated by $L :=$ i-eigenspace of $\mathcal{J}$ under the Clifford action $TX\oplus T^*X \curvearrowright \wedge^\bullet X$ and $U^{n-k} := \wedge^k \bar L \cdot \rho$	$(\wedge^\bullet X, d, d^\wedge)$ bidiff $\mathbb{Z}$-graded vect sp ($d^\wedge := [d, -\iota_{\omega^{-1}}]$) $\phi := \exp(i\omega)\exp\left(\frac{\wedge}{2i}\right)$ yields: $\wedge^k X \xrightarrow{\simeq} U^n$ $d \longmapsto \bar\partial_{\mathcal{J}}$ $d^\wedge \longmapsto 2i\,\partial_{\mathcal{J}}$
THM (NEWLANDER-NIRENBERG). $X \underset{loc}{\simeq} \mathbb{C}^n$	THM (GUALTIERI). In a neighbourhood of a regular point, up to B-transform: $X \underset{loc}{\simeq} \mathbb{C}^k \times (\mathbb{R}^{2n-2k}, \omega_{std})$	THM (DARBOUX). $X \underset{loc}{\simeq} (\mathbb{R}^{2n}, \omega_{std})$

CPLX GEOM. $(\wedge^{\bullet,\bullet}X, \partial, \bar\partial)$ double cplx

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC} \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{\bullet,\bullet}_{\bar\partial} \arrow[dr] & H^\bullet_d \arrow[d] & H^{\bullet,\bullet}_{\partial} \arrow[dl] \\
 & H^{\bullet,\bullet}_A &
\end{tikzcd}$$

Hodge theory: [DOLBEAULT] [SCHWEITZER]

GENERALIZED-COMPLEX GEOM. $(U^\bullet, \partial_{\mathcal{J}}, \bar\partial_{\mathcal{J}})$ bg vect sp

$$\begin{tikzcd}
 & GH^\bullet_{BC} := \frac{\ker \partial_{\mathcal{J}} \cap \ker \bar\partial_{\mathcal{J}}}{\operatorname{im} \partial_{\mathcal{J}}\bar\partial_{\mathcal{J}}} \arrow[dl] \arrow[d] \arrow[dr] & \\
GH^\bullet_{\partial_{\mathcal{J}}} \arrow[dr] & GH_d \arrow[d] & GH^\bullet_{\bar\partial_{\mathcal{J}}} \arrow[dl] \\
 & GH^\bullet_A := \frac{\ker \partial_{\mathcal{J}}\bar\partial_{\mathcal{J}}}{\operatorname{im} \partial_{\mathcal{J}} + \operatorname{im} \bar\partial_{\mathcal{J}}} &
\end{tikzcd}$$

Hodge theory: [GUALTIERI]

SYMPLECTIC GEOM. $(\wedge^\bullet X, d, d^\wedge)$ bg-vect sp

$$\begin{tikzcd}
 & H^\bullet_{dd^\wedge} \arrow[dl] \arrow[d] \arrow[dr] & \\
H^\bullet_d \arrow[dr] & & H^\bullet_{d^\wedge} \arrow[dl] \\
 & H^\bullet_{d+d^\wedge} &
\end{tikzcd}$$

Hodge theory: [TSENG-YAU]

THM (dan. – ADRIANO TOMASSINI).

X cpt gen-cplx. Then:

[*] $\dim_{\mathbb{C}} GH^{\bullet}_{BC_J}(X) + \dim_{\mathbb{C}} GH^{\bullet}_{A_J}(X) \geq \dim_{\mathbb{C}} GH^{\bullet}_{\partial_J}(X) + \dim_{\mathbb{C}} GH^{\bullet}_{\bar\partial_J}(X)$

Moreover, the following are equivalent

- X satisfies $\partial_J\bar\partial_J$-<u>Lemma</u>, i.e., $H_{BC_J} \longrightarrow H_{A_J}$ inj
- "=" holds in $(*)$, and the 'associated' Frölicher spectral sequence degenerates at the first page.

<u>COR</u> X cpt cplx. Then:

$$\dim H^{\bullet\bullet}_{BC} + \dim H^{\bullet\bullet}_{A} \geq \dim H^{\bullet\bullet}_{\partial} + \dim H^{\bullet\bullet}_{\bar\partial} \geq 2 \dim H^{\bullet}_{dR}$$

and TFAE:

$$\partial\bar\partial\text{-Lemma} \iff \begin{cases} \sum_{p+q=k}(h^{p,q}_{BC} + h^{p,q}_{A}) = \sum_{p+q=k}(h^{p,q}_{\partial} + h^{p,q}_{\bar\partial}) \\ b_k = \sum_{p+q=k} h^{p,q}_{\bar\partial} = \sum_{p+q=k} h^{p,q}_{\partial} \end{cases}$$

$$\iff \sum_{p+q=k}(h^{p,q}_{BC} + h^{p,q}_{A}) = 2\, b_k$$

<u>COR.</u> X cpt sympl. Then:

$$\dim H^{\bullet}_{dd^\Lambda} + \dim H^{\bullet}_{d+d^\Lambda} \geq 2 \dim H^{\bullet}_{dR}$$

and TFAE:

$$dd^\Lambda\text{-Lemma} \underset{\substack{[\text{MERKULOV}]\\ [\text{CAVALCANTI}]\\ [\text{GUILLEMIN}]}}{\iff} \text{HLC} \underset{\substack{[\text{MATHIEU}]\\ [\text{YAN}]}}{\iff} \text{Brylinski conj } (\text{i.e., } H_{dd^\Lambda} \to H_{dR} \text{ surj})$$

$$\iff h^{k}_{A} + h^{k}_{BC} = 2\, b_k.$$

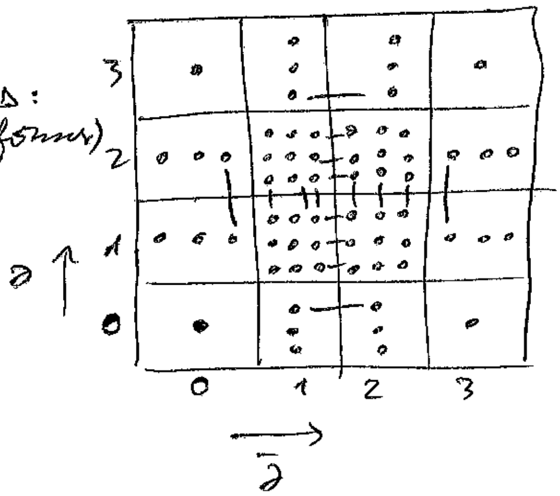
(because Frölicher always degenerates at the first page [FERNANDEZ-IBANEZ-DE LEON]) idea of the proof

$(U^\bullet, \partial_J, \bar\partial_J)$ $\longrightarrow$ $(U^\bullet \otimes \beta^{\mathbb{Z}}, \ \partial_J \otimes \mathrm{id}, \ \bar\partial_J \otimes \beta)$

bi-diff $\mathbb{Z}$-graded alg $\qquad$ bi-diff $\mathbb{Z}^2$-graded alg (double complex)

and then argue as follows:

eg.: (IWASAWA MANIFOLD: left-inv forms)



H_{BC} $\qquad$ H_A

$\#\{\text{ingoing}\} + \#\{\text{outgoing}\}$

$\geq \#\{\text{vertical}\} + \#\{\text{horizontal}\}$

H_∂ $\qquad$ $H_{\bar\partial}$

$\square$

In the cplx case:

$$\Delta^k := \sum_{p+q=k} h^{p,q}_{BC} + \sum_{p+q=k} h^{p,q}_{A} - 2\, b_k \in \mathbb{N}$$

measure the non-cohomolg-Kählerness (= $\partial\bar\partial$-lemma).

For cpt cplx surfaces:

cohom-Kählerness ($\partial\bar\partial$-lemma) $\Longleftrightarrow$ Kähler

$\Updownarrow$ Kodaira also [LAMARI] [BUCHDAHL]

b_1 even

So $\{\Delta^1, \Delta^2\}$ measure non-Kählerness. In fact:

THM (d.an – GEORGES DLOUSSKY – ADRIANO TOMASSINI). $\Delta^1 = 0$.

THM (d.an – ADRIANO TOMASSINI – MISHA VERBITSKY). $\Delta^2 \in \{0, 2\}$.

How to compute explicitly ~~gen~~ cohomologies?

First tool: Hodge theory reduces the problem to PDEs. On ~~mfds with, for~~ homog mfds, we expect the solutions having further symmetries.

Consider solvmfds (nilmfds): cpt quotients $\Gamma\backslash G$ with G connected simply-connected solvable (nilpotent) Lie group and Γ cocpt discrete subgrp.

THM [KASUYA, don-KASUYA]: For some classes of solvmfds ~~that~~ with complex structure (hol parall., Splitting type, left-inv deformations), de Rham, Dolbeault, Bott-Chern cohomologies can be computed just by using a finite-dimensional algebra $(A^{\bullet\bullet}, \partial, \bar\partial) \hookrightarrow (\Lambda^{\bullet\bullet}X, \partial, \bar\partial)$

As regards gen-cplx cohom, our first tool is:

THM (don - SIMONE CALAMAI).
Let $\pi : E \to B$ fibre-bdl of gen-cplx mfds,
with: symplectic fibre F, $\dim F = N$;
complex base B,
Then we have a Leray spectral sequence

$$E_2^{\bullet} = GH_{\bar\partial}(B; \underline{GH}^{N-\bullet}_{\bar\partial}(F)) \Longrightarrow GH_{\bar\partial}(E),$$

where $\underline{GH}^{\bullet}_{\bar\partial}(F) = \bigcup_{b \in B} GH^{\bullet}_{\bar\partial}(\pi^{-1}(b))$, and $N = \dim F$.

proof: $\mathcal{F}^r \Lambda E := \bigoplus_{\ell \geq r} \Lambda(B; U_F^{N-\ell})$

$\rightsquigarrow$ $E_0^r = \Lambda(B; U_F^{N-r})$ with differential δ_0 induced by $\bar\partial_F$

$\rightsquigarrow$ $E_1^r \simeq \Lambda(B; GH_F^{N-r})$ with differential δ_1 induced by $\bar\partial_B$

$\rightsquigarrow$ $E_2^r \simeq GH_{\bar\partial}(B; \underline{GH}_F^{N-r})$.

$\square$

As a corollary, we get a Poincaré-Lemma for gen-cplx structures:

COR. (don - SIMONE CALAMAI).
Any regular point of type k of a gen-cplx mfd of dim $2n$ has a neighbourhood U st

$$GH_{\bar\partial}^{\ell}(U) = \{0\}, \quad \forall\, \ell \neq n-k.$$

proof: By gen-Darboux:

$$(U, \mathcal{J}) \simeq \underset{\text{cplx type}}{(U_1, \mathcal{J}_1)} \times \underset{\text{sympl type}}{(U_2, \mathcal{J}_2)}.$$

By Leray:

$$E_2^r = GH_{\bar\partial}(U_1; \underline{GH}_{\bar\partial}^{(2n-2k)-r}(U_2)) = GH_{\bar\partial}(U_1; H_{dR}^{(n-k)-[(2n-2k)-r]}(U_2))$$

By ~~sympl~~ Poincaré (on U_2): $= GH_{\bar\partial}(U_1; H_{dR}^{r-(n-k)}(U_2)) \Rightarrow GH_{\bar\partial}(U)$

$$E_2^r = 0 \qquad \forall\, r \neq (n-k).$$

Therefore:

$$GH_{\bar\partial}^{\ell}(U) = \frac{F^{n-k} GH_{\bar\partial}^{\ell}(U)}{F^{n-k+1} GH_{\bar\partial}^{\ell}(U)} \Leftarrow \left(E_2^{n-k}\right)^{\ell} = GH_{\bar\partial}^{\ell-(n-k)}(U_1).$$

$\square$

RMK. Note:

- cplx case $\to k = n \to GH_{\bar\partial}^{\ell}(\Delta) = \bigoplus_{p-q=\ell} H_{\bar\partial}^{p,q}(\Delta) = 0, \forall \ell \neq 0$
- sympl case $\to k = 0 \to GH_{\bar\partial}^{\ell}(\Delta) = H_{dR}^{n-\ell}(\Delta) = 0, \forall \ell \neq n.$

Another COR. regards nilmfds:

COR. (Angella – SIMONS – CALAMAI).

Let $X = \Gamma \backslash G$ nilmfd with left-inv gen-cplx structure. ~~Suppose it is suitable~~ ~~For~~ Consider the fibration

$$\begin{array}{ccc} \underset{\text{sympl}}{F} & \hookrightarrow & \cancel{X} \\ & & \downarrow \\ & & \underset{\text{cplx}}{B} \end{array}$$

We assume that the fibration is compatible with the rational structure, and ~~that~~ the Dolbeault cohom of B can be computed by left-inv forms (e.g., the induced left-inv cplx structure on B is hol-par, or Abelian, or nilp, or rational, ...).

Then the gen-cohomologies of X are computed by means of just left-inv forms:

$$GH^{\bullet}_{\bar\partial}(\mathfrak{g}) \xrightarrow{\simeq} GH^{\bullet}_{\bar\partial}(X), \qquad GH^{\bullet}_{BC}(\mathfrak{g}) \xrightarrow{\simeq} GH^{\bullet}_{BC}(X).$$

proof.

$$\begin{array}{ccc} E_2^{r} \simeq GH_{\bar\partial}(B; \underline{GH}^{N-r}_{\bar\partial}(F)) & \Longrightarrow & GH_{\bar\partial}(X) \\ \| & & \\ GH_{\bar\partial}(B) \otimes GH^{N-r}_{\bar\partial}(F) & & \\ \uparrow\simeq \qquad \uparrow\simeq & & \\ GH_{\bar\partial}(\mathfrak{b}) \otimes GH_{\bar\partial}(\mathfrak{f}) & \Longrightarrow & GH_{\bar\partial}(\mathfrak{g}) \end{array}$$

$\square$

Hence, we can compute explicit examples.

The Iwasawa mfd is a 6-dim nilmfd. It admits both a hol-parall left-inv cplx struct, J_{hol}, and an Abelian left-inv cplx struct, J_{ab}. They are disconnected as cplx structures. But [CAVALCANTI-GUALTIERI] proved that they are connected by a path of gen-cplx structures.

THM (dan-SIMONE CALAMAI-ADELA LATORRE). The ~~...~~ curve of gen-cplx structures connecting J_{hol} and J_{ab} satisfies the property

$$GH^{\bullet}_{BC}(X) \longrightarrow GH_{dR} \text{ is surjective.}$$

(This property is weaker than $\partial\bar{\partial}$-lemma. It is equivalent for symplectic.)

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