Cohomologies on complex manifolds

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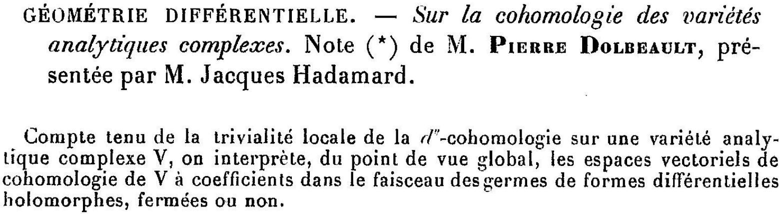
GÉOMÉTRIE DIFFÉRENTIELLE. — *Sur la cohomologie des variétés analytiques complexes.* Note (*) de M. **Pierre Dolbeault**, présentée par M. Jacques Hadamard.

Compte tenu de la trivialité locale de la d''-cohomologie sur une variété analytique complexe V, on interprète, du point de vue global, les espaces vectoriels de cohomologie de V à coefficients dans le faisceau des germes de formes différentielles holomorphes, fermées ou non.

Introduction, ii

It was 50s. . . , ii

Complex geometry encoded in global invariants:

2. Théorème 1. — *Pour tous entiers p, $q \geqq 0$, l'espace vectoriel* $H^q(V, \Omega^p)$ *est canoniquement isomorphe au sous-espace* $H^{p,q}(V)$ *des éléments de type* (p, q) *de la* d''*-cohomologie des courants (resp. des formes différentielles* C^∞*).*

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- Algebraic struct induced by differential algebra $\left(\wedge^{\bullet,\bullet}X, \overline{\partial}, \wedge\right)$.

- Relation with topological informations.

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- Relation with topological informations.

  Sur une variété compacte V de type kählérien,

  THÉORÈME 3. — *L'espace de cohomologie $\mathscr{H}$(V) d'une variété compacte V de type kählérien est somme directe des espaces $\mathscr{H}^{a,b}$(V).*

  A. Weil, *Introduction à l'etude des variétés kählériennes*, Hermann, Paris, 1958.

On a complex (possibly non-Kähler) manifold:

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THEOREM 3. *The Dolbeault groups $H^p(M, \Omega^q)$ form the term E_1 of a spectral sequence, whose term E_∞ is the associated graded C-module of the conveniently filtered de Rham groups. The spectral sequence is stationary after a finite number of steps, and $E_\infty = E_N$ for N sufficiently large.*

A. Frölicher, Relations between the cohomology groups of Dolbeault and topological invariants, *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 641–644.

Interest on non-Kähler manifold since 70s:

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SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

W. P. THURSTON

ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

K. Kodaira, On the structure of compact complex analytic surfaces. I, *Amer. J. Math.* 86 (1964), 751–798.

W. P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* 55 (1976), no. 2, 467–468.

Bott-Chern and Aeppli cohomologies for complex manifolds:

In other words, if we define $\hat{H}^k(X)$ by:

$$\hat{H}^k(X) = A^{k,k}(X) \cap \mathrm{Ker}\,(d)/dd^c A^{k-1,k-1}(X)$$

R. Bott, S. S. Chern, Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections, *Acta Math.* 114 (1965), 71–112.

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$\rightsquigarrow$ they provide bridges between de Rham and Dolbeault cohomologies, allowing their comparison

Aim:
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study the algebra of Bott-Chern cohomology, and its relation with de Rham cohomology:

- use Bott-Chern as degree of "non-Kählerness"...
- ...in order to characterize $\partial\overline{\partial}$-Lemma;
- develop techniques for computations on special classes of manifolds.

Consider the double complex

$$\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$$

associated to a cplx mfd X

Cohomologies of complex manifolds, ii
double complex of forms, ii

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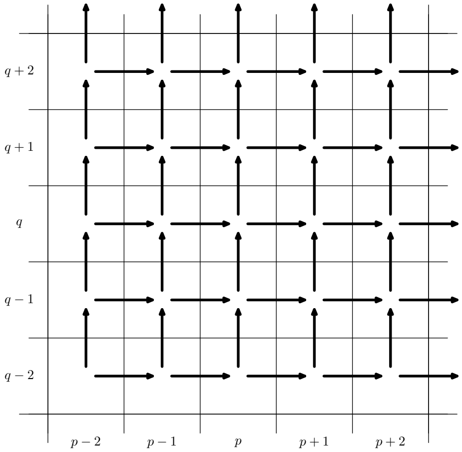
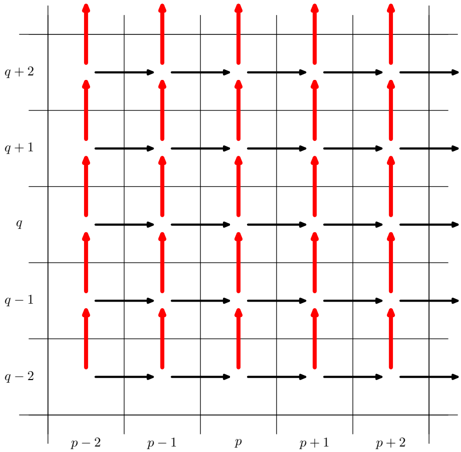


Cohomologies of complex manifolds, iii
Dolbeault cohomology, i

$$H_{\overline{\partial}}^{\bullet,\bullet}(X) := \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}$$



P. Dolbeault, Sur la cohomologie des variétés analytiques complexes, *C. R. Acad. Sci. Paris* 236 (1953), 175–177.

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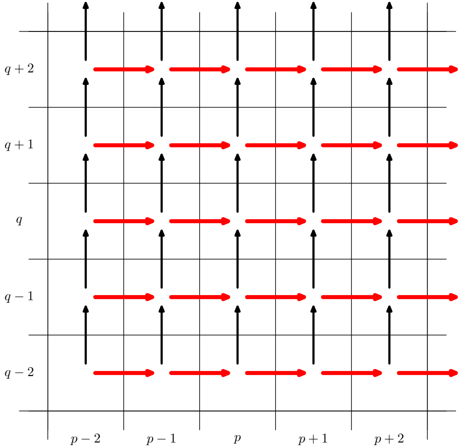

In the Frölicher spectral sequence

$$H^{\bullet,\bullet}_{\bar{\partial}}(X) \Longrightarrow H^{\bullet}_{dR}(X;\mathbb{C})$$

the Dolbeault cohom plays the role of approximation of de Rham.

A. Frölicher, Relations between the cohomology groups of Dolbeault and topological invariants, *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 641–644.

Cohomologies of complex manifolds, vi
Dolbeault cohomology, iv

In the Frölicher spectral sequence

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \Longrightarrow H^{\bullet}_{dR}(X;\mathbb{C})$$

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As a consequence, the Frölicher inequality holds:

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;\geq\; \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})\,.$$

A. Frölicher, Relations between the cohomology groups of Dolbeault and topological invariants, *Proc. Nat. Acad. Sci. U.S.A.* 41 (1955), 641–644.

In general, there is no natural map between Dolb and de Rham:

Cohomologies of complex manifolds, viii

Bott-Chern and Aeppli cohomologies, ii

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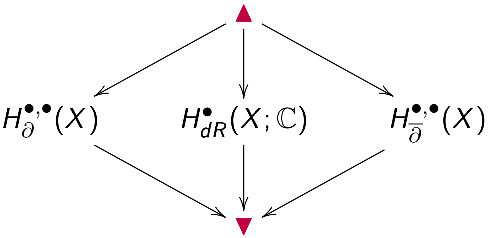

Cohomologies of complex manifolds, ix

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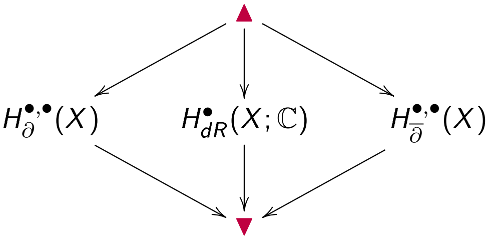
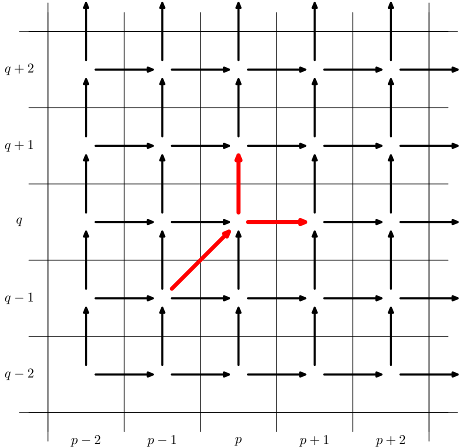


The bridges are provided by Bott-Chern and Aeppli cohomologies.

Cohomologies of complex manifolds, x
Bott-Chern and Aeppli cohomologies, iv

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\operatorname{im} \partial\overline{\partial}}$$



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Cohomologies of complex manifolds, xi

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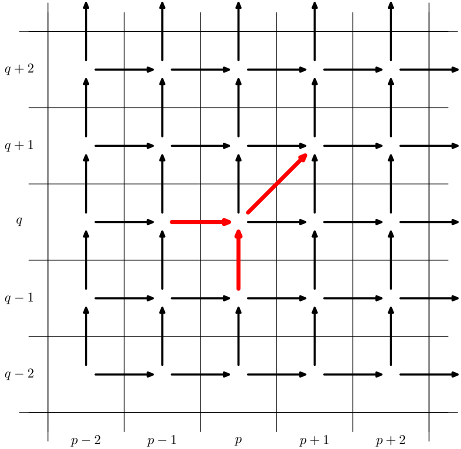

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Cohomological properties of non-Kähler manifolds, i

cohomologies of complex manifolds, i

On cplx mfds, identity induces natural maps

$$
\begin{array}{ccccc}
 & & H^{\bullet,\bullet}_{BC}(X) & & \\
 & \swarrow & \downarrow & \searrow & \\
H^{\bullet,\bullet}_{\partial}(X) & & H^{\bullet}_{dR}(X;\mathbb{C}) & & H^{\bullet,\bullet}_{\overline{\partial}}(X) \\
 & \searrow & \downarrow & \swarrow & \\
 & & H^{\bullet,\bullet}_{A}(X) & &
\end{array}
$$

By def, a cpt cplx mfd satisfies *$\partial\overline{\partial}$-Lemma* if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact too

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Cohomological properties of non-Kähler manifolds, iii
cohomologies of complex manifolds, iii

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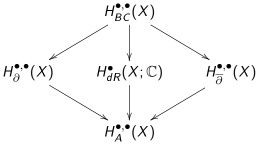

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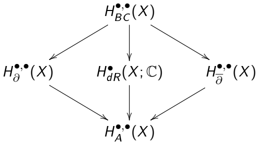

- While compact Kähler mfds satisfy the $\partial\overline{\partial}$-Lemma, . . .

Cohomological properties of non-Kähler manifolds, v
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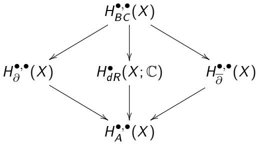
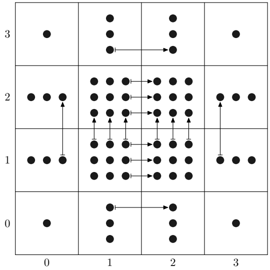


- While compact Kähler mfds satisfy the $\partial\overline{\partial}$-Lemma, ...
- ... Bott-Chern cohomology may supply further informations on the geometry of non-Kähler manifolds.

Cohomological properties of non-Kähler manifolds, vi

inequality *à la* Frölicher for Bott-Chern cohomology, i



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Cohomological properties of non-Kähler manifolds, vii

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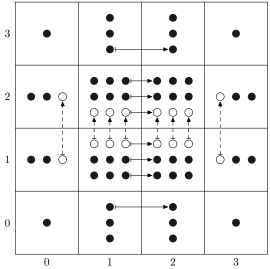
Dolbeault cohomology cares only about horizontal arrows, as Bott-Chern cares only about ingoing arrows, and, dually, Aeppli cares only about outgoing arrows.



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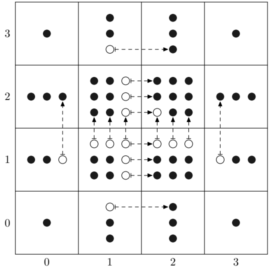

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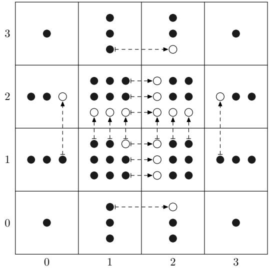

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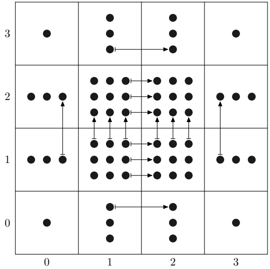
$$\sharp\{\text{ingoing}\} + \sharp\{\text{outgoing}\} \geq \sharp\{\text{horizontal}\} + \sharp\{\text{vertical}\}$$

one gets:



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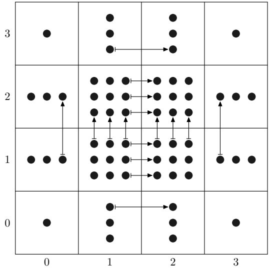

Thm (—, A. Tomassini)

X cpt cplx mfd. The following inequality à la Frölicher holds:

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \geq 2 \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) .$$

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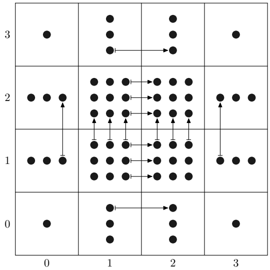

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Furthermore, the equality characterizes the $\partial\overline{\partial}$-Lemma.

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For cpt cplx mfd:

$\Delta^k = 0$ for any k $\Leftrightarrow$ $\partial\overline{\partial}$-Lemma (= cohomologically-Kähler)

(where: $\Delta^k := h^k_{BC} + h^k_A - 2\, b_k \in \mathbb{N}$).

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In fact, non-Kählerness is measured by just $\frac{1}{2}\,\Delta^2 \in \mathbb{N}$.

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is stable for small deformations. Then:

Cor (Voisin; Wu; Tomasiello; —, A. Tomassini)

The property of satisfying the $\partial\overline{\partial}$-Lemma is open under deformations.

Problem:

what happens for limits?

If J_t satisfies $\partial\overline{\partial}$-Lem for any $t \neq 0$, does J_0 satisfy $\partial\overline{\partial}$-Lem, too?

We need tools for investigating explicit examples...

X compact cplx mfd. We want to compute $H^{\bullet,\bullet}_{BC}(X)$.

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For some classes of homogeneous mfds, the solutions of this system may have further symmetries, which reduce to the study of Δ_{BC} on a smaller space. If this space is finite-dim, we are reduced to solve a linear system.

In other words, we would like to reduce the study to a *$H_\sharp$-model*, that is, a sub-algebra

$$\iota\colon\ (M^{\bullet,\bullet},\, \partial,\, \overline{\partial}) \hookrightarrow (\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial})$$

such that $H_\sharp(\iota)$ isomorphism, where $\sharp \in \{dR, \overline{\partial}, \partial, BC, A\}$.

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We are interested in *$H_\sharp$-computable* cplx mfds, that is, admitting a $H_\sharp$-model being finite-dimensional as a vector space.

Thm (Nomizu)

$X = \Gamma\backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

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Then it is H_{dR}-computable.

More precisely, the finite-dimensional sub-space of forms being invariant for the left-action $G \curvearrowright X$ is a H_{dR}-model.

$$
\begin{array}{ccc}
\mathbb{T}^{j_0} & \hookrightarrow & X = \Gamma\backslash G \\
 & & \downarrow \\
\mathbb{T}^{j_1} & \hookrightarrow & X_1 \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\mathbb{T}^{j_k} & \hookrightarrow & X_k \\
 & & \downarrow \\
 & & \mathbb{T}^{j_{k+1}}
\end{array}
$$

Thm (Nomizu)

$X = \Gamma\backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

Then it is H_{dR}-computable.

More precisely, the finite-dimensional sub-space of forms being invariant for the left-action $G \curvearrowright X$ is a H_{dR}-model.

Thm (Nomizu; Console and Fino; —; *et al.*)

$X = \Gamma \backslash G$ *nilmanifold*

K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math. (2)* 59 (1954), no. 3, 531–538.

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—, The cohomologies of the Iwasawa manifold and of its small deformations, *J. Geom. Anal.* 23 (2013), no. 3, 1355-1378.

Cohomological properties of non-Kähler manifolds, xxxii

techniques of computations — nilmanifolds, xii

Thm (Nomizu; Console and Fino; —; *et al.*)

$X = \Gamma \backslash G$ nilmanifold, endowed with a "suitable" left-invariant cplx structure.

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Thm (Nomizu; Console and Fino; —; *et al.*)

$X = \Gamma \backslash G$ nilmanifold, endowed with a "suitable" left-invariant cplx structure.

Then:

- *de Rham cohom* (*Nomizu*)
- *Dolbeault cohom* (*Sakane, Cordero, Fernández, Gray, Ugarte, Console, Fino, Rollenske*)
- *Bott-Chern cohom* (*—*)

can be computed by considering only left-invariant forms.

K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math. (2)* 59 (1954), no. 3, 531–538.

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Iwasawa manifold:

$$\mathbb{I}_3 := (\mathbb{Z}[i])^3 \Big\backslash \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^3) \right\}$$

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- holomorphically-parallelizable nilmanifold
- left-inv co-frame for $\left(T^{1,0}\mathbb{I}_3\right)^*$:

$$\left\{ \varphi^1 := \mathrm{d}\, z^1, \quad \varphi^2 := \mathrm{d}\, z^2, \quad \varphi^3 := \mathrm{d}\, z^3 - z^1\, \mathrm{d}\, z^2 \right\}$$

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- structure equations:

$$\begin{cases} \mathrm{d}\,\varphi^1 = 0 \\ \mathrm{d}\,\varphi^2 = 0 \\ \mathrm{d}\,\varphi^3 = -\varphi^1 \wedge \varphi^2 \end{cases}$$

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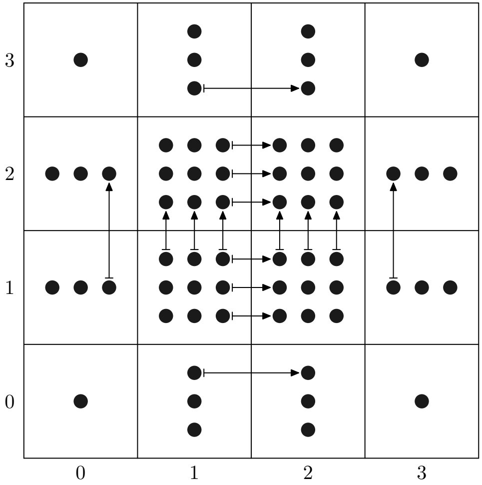
Iwasawa manifold, v



Left-invariant forms provide a finite-dim cohomological-model for the Iwasawa manifold.

Thm (Nakamura)

There exists a locally complete complex-analytic family of complex structures, deformations of $\mathbb{I}_3$, depending on six parameters. They can be divided into three classes according to their Hodge numbers

I. Nakamura, Complex parallelisable manifolds and their small deformations, *J. Differ. Geom.* 10 (1975), no. 1, 85–112.

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Bott-Chern yields a finer classification of Kuranishi space of $\mathbb{I}_3$ (—).

class	$h^1_{\bar\partial}$	h^1_{BC}	$h^2_{\bar\partial}$	h^2_{BC}	$h^3_{\bar\partial}$	h^3_{BC}	$h^4_{\bar\partial}$	h^4_{BC}	$h^5_{\bar\partial}$	h^5_{BC}
(i)	5	4	11	10	14	14	11	12	5	6
(ii.a)	4	4	9	8	12	14	9	11	4	6
(ii.b)	4	4	9	8	12	14	9	10	4	6
(iii.a)	4	4	8	6	10	14	8	11	4	6
(iii.b)	4	4	8	6	10	14	8	10	4	6
	$b_1 = 4$		$b_2 = 8$		$b_3 = 10$		$b_4 = 8$		$b_5 = 4$	

I. Nakamura, Complex parallelisable manifolds and their small deformations, *J. Differ. Geom.* 10 (1975), no. 1, 85–112.

More in general:

any left-invariant complex structure on a 6-dim nilmfd admits a finite-dim cohomological-model (except, perhaps, $\mathfrak{h}_7$)

$\rightsquigarrow$ cohomol classification of 6-dim nilmfds with left-inv cplx struct.

—, M. G. Franzini, F. A. Rossi, Degree of non-Kählerianity for 6-dimensional nilmanifolds, arXiv:1210.0406 [math.DG].

A. Latorre, L. Ugarte, R. Villacampa, On the Bott-Chern cohomology and balanced Hermitian nilmanifolds, arXiv:1210.0395 [math.DG].

Problem:

what about closedness of $\partial\overline{\partial}$-Lemma under limits?

But:

- non-tori nilmanifolds never satisfy $\partial\overline{\partial}$-Lemma (Hasegawa);

K. Hasegawa, Minimal models of nilmanifolds, *Proc. Amer. Math. Soc.* 106 (1989), no. 1, 65–71.

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Therefore:

- consider solvmanifolds *(compact quotients of connected simply-connected solvable Lie groups by co-compact discrete subgroups)*.

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Several tools have been developed for computing cohomologies of solvmanifolds with left-inv cplx structure

A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles, *J. Fac. Sci. Univ. Tokyo Sect. I* 8 (1960), no. 1960, 289–331.

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H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems, *J. Differ. Geom.* 93, (2013), 269–298.

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Several tools have been developed for computing cohomologies of solvmanifolds with left-inv cplx structure, and of their deformations (H. Kasuya; —, H. Kasuya).

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Thanks to these tools:

Thm (—, H. Kasuya)

The property of satisfying the $\partial\overline{\partial}$-Lemma is non-closed under deformations.

—, H. Kasuya, Cohomologies of deformations of solvmanifolds and closedness of some properties, arXiv:1305.6709 [math.CV].

The Lie group

$$\mathbb{C} \ltimes_{\phi} \mathbb{C}^2 \quad \text{dove} \quad \phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} .$$

admits a lattice: the quotient is called Nakamura manifold.

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$$t \, \frac{\partial}{\partial z^1} \otimes \mathrm{d}\, \bar{z}^1 \ .$$

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$\leadsto$ the previous theorems furnish finite-dim sub-complexes to compute Dolbeault and Bott-Chern cohomologies.

$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp}$	Nakamura			deformations		
	dR	$\bar{\partial}$	BC	dR	$\bar{\partial}$	BC
$(0,0)$	1	1	1	1	1	1
$(1,0)$	2	3	1	2	1	1
$(0,1)$		3	1		1	1
$(2,0)$	5	3	3	5	1	1
$(1,1)$		9	7		3	3
$(0,2)$		3	3		1	1
$(3,0)$	8	1	1	8	1	1
$(2,1)$		9	9		3	3
$(1,2)$		9	9		3	3
$(0,3)$		1	1		1	1
$(3,1)$	5	3	3	5	1	1
$(2,2)$		9	11		3	3
$(1,3)$		3	3		1	1
$(3,2)$	2	3	5	2	1	1
$(2,3)$		3	5		1	1
$(3,3)$	1	1	1	1	1	1

- Cplx structure:
  $J\colon TX \xrightarrow{\cong} TX$ satisfying an algebraic condition ($J^2 = -\mathrm{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).

- Cplx structure:
  $J\colon TX \xrightarrow{\simeq} TX$ satisfying an algebraic condition ($J^2 = -\mathrm{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).
- Sympl structure:
  $\omega\colon TX \xrightarrow{\simeq} T^*X$ satisfying an algebraic condition (ω non-deg 2-form) and an analytic condition ($\mathrm{d}\,\omega = 0$).

Hence, consider the bundle $TX \oplus T^*X$.

N. J. Hitchin, Generalized Calabi-Yau manifolds, *Q. J. Math.* **54** (2003), no. 3, 281–308.

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Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X + \xi | Y + \eta \rangle = \frac{1}{2} \left(\iota_X \eta + \iota_Y \xi \right)$.

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Mimicking the def of cplx and sympl structures:

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Generalized-complex geometry, vi
generalized-complex structures, vi

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Mimicking the def of cplx and sympl structures:

a generalized-complex structure on a $2n$-dim mfd X is a

$$\mathcal{J} \colon TX \oplus T^*X \to TX \oplus T^*X$$

such that $\mathcal{J}^2 = -\mathrm{id}_{TX \oplus T^*X}$, being orthogonal wrt $\langle - | = \rangle$, and satisfying an integrability condition.

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Generalized-complex geometry, x

cohomological properties of symplectic manifolds, i

This explains the parallel between the cplx and sympl contexts:

L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: I. II, *J. Differ. Geom.* 91 (2012), no. 3, 383–416, 417–443.

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as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.

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as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.
Define the cohomologies

$$H^{\bullet}_{BC,\omega}(X) := \frac{\ker \mathrm{d} \cap \ker \mathrm{d}^{\Lambda}}{\operatorname{im} \mathrm{d}\,\mathrm{d}^{\Lambda}} \quad \text{and} \quad H^{\bullet}_{A,\omega}(X) := \frac{\ker \mathrm{d}\,\mathrm{d}^{\Lambda}}{\operatorname{im} \mathrm{d} + \operatorname{im} \mathrm{d}^{\Lambda}} .$$

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Thm (Merkulov; Guillemin; Cavalcanti; —, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd.

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Generalized-complex geometry, xv
cohomological properties of symplectic manifolds, vi

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Let X be a $2n$-dim cpt symplectic mfd. Then, for any k,

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \;\geq\; 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) \, .$$

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Furthermore, the following are equivalent:

- *X satisfies $\mathrm{d}\,\mathrm{d}^{\Lambda}$-Lemma (i.e., Bott-Chern and de Rham cohom are natur isom);*
- *X satisfies Hard Lefschetz Cond (i.e., $[\omega^k]\colon H^{n-k}_{dR}(X) \to H^{n+k}_{dR}(X)$ isom $\forall k$);*
- *equality $\dim H^k_{BC,\omega}(X) + \dim H^k_{A,\omega} = 2\,\dim H^k_{dR}(X;\mathbb{R})$ holds for any k.*

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Joint work with: Adriano Tomassini, Hisashi Kasuya, Federico A. Rossi, Maria Giovanna Franzini, Simone Calamai, Weiyi Zhang, Georges Dloussky.

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