Cohomologies on complex manifolds

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Introduction, i
It was 50s... i


Compte tenu de la trivialité locale de la $d^c$-cohomologie sur une variété analytique complexe V, on interprète, du point de vue global, les espaces vectoriels de cohomologie de V à coefficients dans le faisceau des germes de formes différentielles holomorphes, fermées ou non.
**Complex geometry encoded in global invariants:**

2. **Théorème 1.** — Pour tous entiers $p, q \geq 0$, l'espace vectoriel $H^q(V, \Omega^p)$ est canoniquement isomorphe au sous-espace $H^{p,q}(V)$ des éléments de type $(p, q)$ de la $d$-cohomologie des courants (resp. des formes différentielles $C^*$).

**What informations in $\overline{\partial}$-cohomology?**
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↬ Algebraic struct induced by differential algebra $(\wedge^\bullet \cdot X, \overline{\partial}, \wedge)$.

↬ Relation with topological informations.


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What informations in $\overline{\partial}$-cohomology?

$q$ Algebraic struct induced by differential algebra $(\wedge^\bullet X, \overline{\partial}, \wedge)$.


$q$ Relation with topological informations.

Sur une variété compacte $V$ de type kählerien,

Théorème 3. — L'espace de cohomologie $\mathcal{H}(V)$ d'une variété compacte $V$
de type kählerien est somme directe des espaces $\mathcal{H}^{ab}(V)$.


On a complex (possibly non-Kähler) manifold:
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**Theorem 3.** The Dolbeault groups $H^p(M, \Omega)$ form the term $E_1$ of a spectral sequence, whose term $E_\infty$ is the associated graded $C$-module of the conveniently filtered de Rham groups. The spectral sequence is stationary after a finite number of steps, and $E_\infty = E_N$ for $N$ sufficiently large.


Interest on non-Kähler manifold since 70s:
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SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

W. P. THURSTON

ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.


Bott-Chern and Aeppli cohomologies for complex manifolds:

In other words, if we define $\check{H}^k(X)$ by:

$\check{H}^k(X) = A^{k,k}(X) \cap \text{Ker } (d) \cap d^{k-1}A^{k-1,k-1}(X)$


Bott-Chern and Aeppli cohomologies for complex manifolds:

In other words, if we define $\hat{H}^k(X)$ by:

$$\hat{H}^k(X) = A^{*,k}(X) \cap \ker (d) \cap d\Omega^{k-1,*}(X)$$


they provide bridges between de Rham and Dolbeault cohomologies, allowing their comparison.

Aim:

study the algebra of Bott-Chern cohomology, and its relation with de Rham cohomology:
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- use Bott-Chern as degree of "non-Kählerness"

- in order to characterize $\partial \bar{\partial}$-Lemma;
**Aim:**
study the algebra of Bott-Chern cohomology, and its relation with de Rham cohomology:

- use Bott-Chern as degree of "non-Kählerness"…
- …in order to characterize $\partial\bar{\partial}$-Lemma;
- develop techniques for computations on special classes of manifolds.

Consider the double complex associated to a cplx mfd $X$: 

$$(\wedge^\bullet \mathcal{O}_X, \partial, \partial)$$
Consider the double complex

\[(\wedge^{\cdot, \cdot} X, \partial, \overline{\partial})\]

associated to a cplx mfd \(X\)

\[H^{\cdot, \cdot}_\overline{\partial}(X) := \frac{\ker \overline{\partial}}{\text{im} \overline{\partial}}\]
In the Frölicher spectral sequence

\[ H^p_q(X) \rightarrow H^p_{dR}(X; \mathbb{C}) \]

the Dolbeault cohom plays the role of approximation of de Rham.
In the Frölicher spectral sequence

$$H^\bullet\bullet(\mathcal{X}) \Longrightarrow H^\bullet_{dR}(\mathcal{X}; \mathbb{C})$$

the Dolbeault cohom plays the role of approximation of de Rham.

As a consequence, the Frölicher inequality holds:

$$\sum_{p+q=k} \dim \mathbb{C} H^{p,q}_{\partial}(\mathcal{X}) \geq \dim \mathbb{C} H^k_{dR}(\mathcal{X}; \mathbb{C}).$$


In general, there is no natural map between Dolb and de Rham:
In general, there is no natural map between Dolb and de Rham:
we would like to have a “bridge” between them.

The bridges are provided by Bott-Chern and Aeppli cohomologies.
Cohomologies of complex manifolds, x
Bott-Chern and Aeppli cohomologies, iv

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\operatorname{im} \overline{\partial \partial}}$$


Cohomologies of complex manifolds, xi
Bott-Chern and Aeppli cohomologies, v

$$H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \overline{\partial}}{\operatorname{im} \partial + \operatorname{im} \overline{\partial}}$$

On cplx mfds, identity induces natural maps

\[ H_{\bar{\partial}}^\bullet(X) \leftarrow H_{\partial}^\bullet(X) \rightarrow H_{dR}^\bullet(X; \mathbb{C}) \rightarrow H_{\bar{\partial}}^\bullet(X) \]

\[ H_{\partial}^\bullet(X) \leftarrow H_{dR}^\bullet(X; \mathbb{C}) \rightarrow H_{\bar{\partial}}^\bullet(X) \rightarrow H_{\partial}^\bullet(X) \]

By def, a cpt cplx mfd satisfies \( \partial\bar{\partial}\text{-Lemma} \) if every \( \partial \)-closed \( \bar{\partial} \)-closed d-exact form is \( \partial\bar{\partial} \)-exact too
By def, a cpt cplx mfd satisfies $\partial\bar{\partial}$-Lemma if every $\partial$-closed $\bar{\partial}$-closed d-exact form is $\partial\bar{\partial}$-exact too, equivalently, if all the above maps are isomorphisms.

While compact Kähler mfds satisfy the $\partial\bar{\partial}$-Lemma, . . .
By def, a cpt cplx mfd satisfies $\partial\bar{\partial}$-Lemma if every $\partial$-closed $\bar{\partial}$-closed $d$-exact form is $\partial\bar{\partial}$-exact too, equivalently, if all the above maps are isomorphisms.

- While compact Kähler mfds satisfy the $\partial\bar{\partial}$-Lemma, . . .
- . . . Bott-Chern cohomology may supply further informations on the geometry of non-Kähler manifolds.

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**Cohomological properties of non-Kähler manifolds, vi**

inequality à la Frölicher for Bott-Chern cohomology, i

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Dolbeault cohomology cares only about horizontal arrows, as Bott-Chern cares only about ingoing arrows, and, dually, Aeppli cares only about outgoing arrows.

Thm (A. Tomassini) X cpt cplx mfd. The following inequality à la Frölicher holds:

$$\sum_{p+q=k} \left( \dim \mathcal{C}H_{p,q}^{BC}(X) + \dim \mathcal{C}H_{p,q}^{A}(X) \right) \geq 2 \dim \mathcal{C}H_{k,dR}(X; \mathbb{C})$$

Furthermore, the equality characterizes the $\partial \bar{\partial}$-Lemma.

Dolbeault cohomology cares only about horizontal arrows, as Bott-Chern cares only about ingoing arrows, and, dually, Aeppli cares only about outgoing arrows.

Since
\[ \#\{\text{ingoing}\} + \#\{\text{outgoing}\} \geq \#\{\text{horizontal}\} + \#\{\text{vertical}\} \]

one gets:
Dolbeault cohomology cares only about horizontal arrows, as Bott-Chern cares only about ingoing arrows, and, dually, Aeppli cares only about outgoing arrows. Since
$$\# \{\text{ingoing}\} + \# \{\text{outgoing}\} \geq \# \{\text{horizontal}\} + \# \{\text{vertical}\}$$

one gets:

Thm (—, A. Tomassini)

$X$ cpt cplx mfd. The following inequality à la Frölicher holds:

$$\sum_{p+q=k} (\dim \mathbb{C} H_{BC}^{p,q}(X) + \dim \mathbb{C} H_A^{p,q}(X)) \geq 2 \dim \mathbb{C} H_{dR}^k(X; \mathbb{C}) .$$

Furthermore, the equality characterizes the $\partial \bar{\partial}$-Lemma.

For cpt cplx mfd:

\[ \Delta^k = 0 \text{ for any } k \iff \partial\bar{\partial}-\text{Lemma (}= \text{cohomologically-Kähler)} \]

(where: \( \Delta^k := h_{BC}^k + h_A^k - 2b_k \in \mathbb{N} \)).

For cpt cplx surfaces:

\[ \text{Kähler} \iff b_1 \text{ even} \iff \text{cohom-Kähler} \]
For cpt cplx mfd:

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Forcpt cplx surfaces:

\text{Kähler } \iff \text{b}_1 \text{ even } \iff \text{cohom-Kähler}

\text{hence } \Delta^1 \text{ and } \Delta^2 \text{ measure just Kählerness.}

\[ \text{In fact, non-Kählerness is measured by just } \frac{1}{2} \Delta^2 \in \mathbb{N}. \]
By Hodge theory, \( \dim \mathbb{C} H^p_{BC} \) and \( \dim \mathbb{C} H^p_A \) are upper-semi-continuous for deformations of the complex structure.

\[
\sum_{p+q=k} (\dim \mathbb{C} H^p_{BC}(X) + \dim \mathbb{C} H^p_A(X)) = 2 \dim \mathbb{C} H^k_{dR}(X; \mathbb{C})
\]

is stable for small deformations.
By Hodge theory, \( \dim \mathbb{C} H^{p,q}_{BC} \) and \( \dim \mathbb{C} H^{p,q}_A \) are upper-semi-continuous for deformations of the complex structure. Hence the equality

\[
\sum_{p+q=k} (\dim \mathbb{C} H^{p,q}_{BC}(X) + \dim \mathbb{C} H^{p,q}_A(X)) = 2 \dim \mathbb{C} H^{k}_{\text{dR}}(X; \mathbb{C})
\]

is stable for small deformations. Then:

**Cor (Voisin; Wu; Tomasiello; —, A. Tomassini)**

The property of satisfying the \( \partial \bar{\partial} \)-Lemma is **open** under deformations.

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**Problem:**

what happens for limits?

If \( J_t \) satisfies \( \partial \bar{\partial} \)-Lem for any \( t \neq 0 \), does \( J_0 \) satisfy \( \partial \bar{\partial} \)-Lem, too?

*We need tools for investigating explicit examples.*
$X$ compact cplx mfd. We want to compute $H_{BC}^{\bullet \bullet}(X)$.

Hodge theory reduces the probl to a pde system
$X$ compact cplx mfd. We want to compute $H_{BC}^{\bullet\bullet}(X)$.

**Hodge theory** reduces the probl to a pde system: fixed $g$ Hermitian metric, there is a 4th order elliptic differential operator $\Delta_{BC}$ s.t.

\[
H_{BC}^{\bullet\bullet}(X) \simeq \ker \Delta_{BC} = \left\{ u \in \wedge^{p,q}X : \partial u = \bar{\partial} u = (\partial \bar{\partial})^* u \right\} .
\]

For some classes of homogeneous mfds, the solutions of this system may have further symmetries, which reduce to the study of $\Delta_{BC}$ on a smaller space.
X compact cplx mfd. We want to compute $H_{BC}^{\bullet,\bullet}(X)$.

Hodge theory reduces the probl to a pde system: fixed $g$ Hermitian metric, there is a 4th order elliptic differential operator $\Delta_{BC}$ s.t.

$$H_{BC}^{\bullet,\bullet}(X) \cong \ker \Delta_{BC} = \left\{ u \in \wedge^{p,q}X : \partial u = \bar{\partial} u = (\partial \bar{\partial})^* u \right\}.$$

For some classes of homogeneous mfds, the solutions of this system may have further symmetries, which reduce to the study of $\Delta_{BC}$ on a smaller space. If this space is finite-dim, we are reduced to solve a linear system.

In other words, we would like to reduce the study to a $H^{\sharp}$-model, that is, a sub-algebra

$$\iota : (M^{\bullet,\bullet}, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet,\bullet}X, \partial, \bar{\partial})$$

such that $H^{\sharp}(\iota)$ isomorphism, where $\sharp \in \{dR, \bar{\partial}, \partial, BC, A\}$. 
In other words, we would like to reduce the study to a $H_\#\text{-model}$, that is, a sub-algebra

$$\iota : (M^{\bullet, \bullet}, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet, \bullet}X, \partial, \bar{\partial})$$

such that $H_\#(\iota)$ isomorphism, where $\# \in \{dR, \bar{\partial}, \partial, BC, A\}$.

We are interested in $H_\#\text{-computable}$ cplx mfds, that is, admitting a $H_\#\text{-model}$ being finite-dimensional as a vector space.

\[\text{Thm (Nomizu)}\]

$X = \Gamma \backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups $G$ by co-compact discrete subgroups $\Gamma$).
Thm (Nomizu)

\[ X = \Gamma \backslash G \] nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups \( G \) by co-compact discrete subgroups \( \Gamma \)).

Then it is \( H_{dR} \)-computable.

More precisely, the finite-dimensional sub-space of forms being invariant for the left-action \( G \curvearrowleft X \) is a \( H_{dR} \)-model.
Thm (Nomizu; Console and Fino; —; et al.)

\[ X = \Gamma \backslash G \text{ nilmanifold} \]

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Thm (Nomizu; Console and Fino; —; et al.)

\[ X = \Gamma \backslash G \text{ nilmanifold}, \text{ endowed with a “suitable” left-invariant cplx structure.} \]

---


Thm \((\text{Nomizu}; \text{Console and Fino}; —; \text{et al.})\)

\(X = \Gamma \backslash G \text{ nilmanifold}, \) endowed with a “suitable” left-invariant cplx structure.

Then:

- de Rham cohom \((\text{Nomizu})\)
- Dolbeault cohom \((\text{Sakane, Cordero, Fernández, Gray, Ugarte, Console, Fino, Rollenske})\)
- Bott-Chern cohom \((-\text{)}\)

can be computed by considering only left-invariant forms.

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**Iwasawa manifold:**

\[
\mathbb{I}_3 := (\mathbb{Z}[i])^3 \setminus \left\{ \left( \begin{array}{ccc} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{array} \right) \in \text{GL}(\mathbb{C}^3) \right\}
\]
Iwasawa manifold:

\[ I_3 := (\mathbb{Z}[i])^3 \setminus \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}(\mathbb{C}^3) \right\} \]

- holomorphically-parallelizable nilmanifold

- left-inv co-frame for \((T^{1,0}I_3)^*\):
  \[ \{ \varphi^1 := dz^1, \quad \varphi^2 := dz^2, \quad \varphi^3 := dz^3 - z^1 dz^2 \} \]
Iwasawa manifold:

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- holomorphically-parallelizable nilmanifold

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  \[ \{ \varphi^1 := dz^1, \quad \varphi^2 := dz^2, \quad \varphi^3 := dz^3 - z^1 dz^2 \} \]

- structure equations:

\[
\begin{align*}
  d \varphi^1 &= 0 \\
  d \varphi^2 &= 0 \\
  d \varphi^3 &= -\varphi^1 \wedge \varphi^2
\end{align*}
\]

Left-invariant forms provide a finite-dim cohomological-model for the Iwasawa manifold.
Thm (Nakamura)

There exists a locally complete complex-analytic family of complex structures, deformations of $\mathbb{I}_3$, depending on six parameters. They can be divided into three classes according to their Hodge numbers.

Bott-Chern yields a finer classification of Kuranishi space of $\mathbb{I}_3$. 

<table>
<thead>
<tr>
<th>Class</th>
<th>$h_1^{\partial}$</th>
<th>$h_1^{BC}$</th>
<th>$h_2^{\partial}$</th>
<th>$h_2^{BC}$</th>
<th>$h_3^{\partial}$</th>
<th>$h_3^{BC}$</th>
<th>$h_4^{\partial}$</th>
<th>$h_4^{BC}$</th>
<th>$h_5^{\partial}$</th>
<th>$h_5^{BC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>5</td>
<td>4</td>
<td>11</td>
<td>10</td>
<td>14</td>
<td>14</td>
<td>11</td>
<td>12</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>(ii.a)</td>
<td>4</td>
<td>4</td>
<td>9</td>
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<td>(ii.b)</td>
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<td>(iii.b)</td>
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<tr>
<td></td>
<td>$b_1 = 4$</td>
<td>$b_2 = 8$</td>
<td>$b_3 = 10$</td>
<td>$b_4 = 8$</td>
<td>$b_5 = 4$</td>
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</tr>
</tbody>
</table>
More in general:

any left-invariant complex structure on a 6-dim nilmfld admits a finite-dim cohomological-model (except, perhaps, $h_7$)

$\leadsto$ cohomol classification of 6-dim nilmfds with left-inv cplx struct.

Problem:

what about closedness of $\bar{\partial}\partial$-Lemma under limits?

But:

- non-tori nilmanifolds never satisfy $\bar{\partial}\partial$-Lemma \hfill (Hasegawa);
Problem: what about closedness of $\partial\bar{\partial}$-Lemma under limits?

But:
- non-tori nilmanifolds never satisfy $\partial\bar{\partial}$-Lemma (Hasegawa);
- tori are closed (Andreotti and Grauert and Stoll).

Therefore:
- consider solvmanifolds (compact quotients of connected simply-connected solvable Lie groups by co-compact discrete subgroups).

Several tools have been developed for computing cohomologies of solvmanifolds with left-inv cplx structure.
Thanks to these tools:

**Thm (—, H. Kasuya)**

*The property of satisfying the \( \partial \overline{\partial} \)-Lemma is non-closed under deformations.*

The Lie group

\[
\mathbb{C} \ltimes \phi \mathbb{C}^2 \quad \text{dove} \quad \phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.
\]

admits a lattice: the quotient is called **Nakamura manifold**.
The Lie group

$$\mathbb{C} \ltimes \phi \mathbb{C}^2 \quad \text{dove} \quad \phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$ 

admits a lattice: the quotient is called Nakamura manifold. Consider the small deformations in the direction

$$t \frac{\partial}{\partial z^1} \otimes d \bar{z}^1.$$ 

\[\Rightarrow\] the previous theorems furnish finite-dim sub-complexes to compute Dolbeault and Bott-Chern cohomologies.
### Generalized-complex geometry, i
**generalized-complex structures, i**

- **Cplx structure:**
  \[ J: TX \cong TX \] satisfying an **algebraic condition** \( J^2 = -\text{id}_{TX} \)
  and an **analytic condition** (integrability in order to have holomorphic coordinates).

### Table of dimensions and Nakamura deformations

<table>
<thead>
<tr>
<th>( \dim_{C} H^*_\mathbb{C} )</th>
<th>Nakamura ( dR ) ( \bar{\partial} ) BC</th>
<th>deformations ( dR ) ( \bar{\partial} ) BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
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</tbody>
</table>
Generalized-complex geometry, ii

generalized-complex structures, ii

- **Cplx structure:** 
  \( J: TX \cong TX \) satisfying an algebraic condition \( J^2 = -\text{id}_{TX} \) and an analytic condition (integrability in order to have holomorphic coordinates).

- **Sympl structure:** 
  \( \omega: TX \cong T^*X \) satisfying an algebraic condition (\( \omega \) non-deg 2-form) and an analytic condition (\( d\omega = 0 \)).

Hence, consider the bundle \( TX \oplus T^*X \).

---


Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X + \xi | Y + \eta \rangle = \frac{1}{2} (\iota_X \eta + \iota_Y \xi)$.

*Mimicking the def of cplx and sympl structures:*

---


Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X + \xi | Y + \eta \rangle = \frac{1}{2} (\iota_X \eta + \iota_Y \xi)$.

 Mimicking the def of cplx and sympl structures:

 a **generalized-complex structure** on a 2n-dim mfd $X$ is a

 $$\mathcal{J}: TX \oplus T^*X \to TX \oplus T^*X$$

 such that $\mathcal{J}^2 = -\text{id}_{TX \oplus T^*X}$, being orthogonal wrt $\langle -| - \rangle$, and satisfying an integrability condition.

---


Generalized-complex geometry unifies cplx geom and sympl geom:

- **$J$ cplx struct**: then

  \[ J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix} \]

  is generalized-complex;

- **$\omega$ sympl struct**: then

  \[ J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \]

  is generalized-complex.
This explains the parallel between the cplx and sympl contexts:

for a symplectic manifold, consider the operators

\[ d : \bigwedge^\bullet X \to \bigwedge^{\bullet+1} X \quad \text{and} \quad d^\Lambda := [d, -\iota_\omega - 1] : \bigwedge^\bullet X \to \bigwedge^{\bullet-1} X \]

as the counterpart of \( \partial \) and \( \overline{\partial} \) in complex geometry.
This explains the parallel between the cplx and sympl contexts: e.g., for a symplectic manifold, consider the operators
\[
d : \wedge^* X \rightarrow \wedge^{*+1} X \quad \text{and} \quad d^\wedge := [d, -i_\omega] : \wedge^* X \rightarrow \wedge^{*-1} X
\]
as the counterpart of \( \partial \) and \( \overline{\partial} \) in complex geometry.

Define the cohomologies
\[
H_{BC,\omega}^*(X) := \frac{\ker d \cap \ker d^\wedge}{\text{im } d^\wedge} \quad \text{and} \quad H_{A,\omega}^*(X) := \frac{\ker d d^\wedge}{\text{im } d + \text{im } d^\wedge}.
\]


Thm (Merkulov; Guillemin; Cavalcanti; —, A. Tomassini)

Let \( X \) be a 2n-dim cpt symplectic mfd.
Thm (Merkulov; Guillemin; Cavalcanti; —, A. Tomassini)

Let $X$ be a $2n$-dim cpt symplectic mfd. Then, for any $k$,

$$\dim_{\mathbb{R}} H^{k}_{BC,\omega}(X) + \dim_{\mathbb{R}} H^{k}_{A,\omega} \geq 2 \dim_{\mathbb{R}} H^{k}_{dR}(X; \mathbb{R}).$$

Furthermore, the following are equivalent:

- $X$ satisfies $d d^\wedge$-Lemma (i.e., Bott-Chern and de Rham cohom are natur isom);
- $X$ satisfies Hard Lefschetz Cond (i.e., $[\omega^k] : H^{n-k}_{dR}(X) \to H^{n+k}_{dR}(X)$ isom $\forall k$);
- equality $\dim H^{k}_{BC,\omega}(X) + \dim H^{k}_{A,\omega} = 2 \dim H^{k}_{dR}(X; \mathbb{R})$ holds for any $k$.


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