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"SPECIAL METRICS ON COMPLEX MANIFOLDS: ON THE CHERN-YAMABE PROBLEM"

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§ INTRODUCTION

AIM. Search for special (possibly "canonical" - meaning the natural transf. preserves the metric) [Hermitian] metrics on cpt cplx mfds.

What "special metric" means:

- with curvature properties:
  (eg.: constant scalar curvature; ~~Kähler~~ Einstein; extremale; ...)
- with "cohomological" properties generalizing Kähler condition:
  (eg.: Gauduchon; balanced in the sense of Michelsohn; pluriclosed or SKT; ...)
- a combination of the ~~two~~:
  (eg.: Kähler-Einstein; cscK; ...)

Two results ~~to~~ this aim:

THM (GAUDUCHON). ~~Any conformal class~~ Any Hermitian-conformal class on a cpt cplx n-manifold contains a standard metric satisfying $\partial\bar\partial\omega^{n-1}=0$, unique up to homothety.
It is ~~called~~ called Gauduchon metric.

THM (YAMABE; TRUDINGER; AUBIN; SCHOEN). Any conformal class on a cpt differentiable manifold contains a constant scalar curvature metric.

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§ CHERN – YAMABE PROBLEM

We investigate an analogue of the Yamabe problem for cpt cplx manifolds and Herm-conformal structures.

Since, for non-Kähler metrics*, ∇^{LC} does not preserve the cplx structure J, we consider Hermitian connections, that is, connections preserving both the metric and the cplx structure, and ~~then~~ then having torsion.

One restricts to canonical connections in the sense of Gauduchon*: they are a 1-parameter family with prescribed torsion:

$$T_t^{11} = 0 \qquad \text{(where: } B_t := B - B_c,$$
$$B_c := \tfrac{1}{2} g(B_{[\cdot, \dots}, -_{]})^{\uparrow} \quad \text{).}$$

In particular, we consider the Chern connection, corresponding to $t = 1$:

it is the unique Herm. connection with $T^{11} = 0$,

equivalently, the unique connection on $T^{10}X$ preserving the Herm structure and whose $(0,1)$-part coincides with the Cauchy-Riemann operator associated to the holomorphic structure.

Its scalar curvature is given by

$$S^{Ch}(\omega) = \mathrm{tr}_{\omega}\, i \bar{\partial} \partial \log \omega^n .$$

*(Similar argument may possibly work for other canonical connections; we consider Chern conn for the rôle of Chern-Ricci curvature in non-Kähler Calabi-Yau, after the work by, eg, Tosatti and Weinkove.)

More precisely:

Let X be cpt cplx mfd, $\dim_{\mathbb{C}} X = n$.
Fix $\{\omega\}$ conformal class of Hermitian metrics.
We denote by $\eta \in \{\omega\}$ the unique Gauduchon metric in $\{\omega\}$ with $\int_X d\mu_\eta = 1$.

We have a natural action:
$$\mathcal{G}_X(\{\omega\}) := \mathcal{H}\mathrm{Conf}(X; \{\omega\}) \times \mathbb{R}^+ \curvearrowright \{\omega\},$$
where $\mathcal{H}\mathrm{Conf}(X; \{\omega\})$ denotes the space of bihol. automorphisms of X preserving $\{\omega\}$.

We study the moduli space:
$$\mathcal{CY}(X; \{\omega\}) := \left\{ \omega' \in \{\omega\} \;:\; S^{Ch}(\omega') \text{ constant} \right\} \Big/ \mathcal{G}_X(\{\omega\})$$

As a first problem, we have the following analogue to the Yamabe problem:

<u>CHERN-YAMABE PROBLEM</u>. Let X be cpt cplx, let $\{\omega\}$ be a conformal class of Hermitian metrics. Prove
$$\mathcal{CY}(X; \{\omega\}) \neq \emptyset.$$

<u>RMK</u>. By LIU-YANG [arXiv:1404.2481], if the Chern scalar curvature is equal to the scalar curvature (resp., to the J-scalar curvature) of the corresponding Riemannian metric, then the metric is Kähler.

Then, Chern-Yamabe problem goes in different direction wrt both Yamabe pb, and Yamabe pb for alm-Hermitian mfds as studied by del Rio and Simanca.

§ CHERN-YAMABE SEMILINEAR PDE

We ~~give~~ translate the Chern-Yamabe problem in ~~a~~ ~~…~~ a semilinear PDE of Liouville-type.

~~By changing~~ Under conformal transformations, the Chern-scalar curvature changes as:

$$S^{Ch}(\exp(2f/n)\,\omega) = \exp(-2f/n)\left(S^{Ch}(\omega) + \Delta^{Ch}_{\omega} f\right).$$

Here, $\Delta^{Ch}_{\omega} f := (\omega, dd^c f)_{\omega} = 2\, i\, \mathrm{tr}_{\omega}\, \bar\partial\partial f$ denotes the Laplacian associated to the Chern connection.
In terms of the de Rham Laplacian ($\Delta_{d,\omega} = d^*_{\omega} d$), one writes:

$$\Delta^{Ch}_{\omega} f = \Delta_{d,\omega} f + (df \mid \vartheta)_{\omega},$$

where ϑ denotes the Lee 1-form of ω (that is, $d\omega^{n-1} = \vartheta \wedge \omega^{n-1}$).

In particular, note that:

- for Gauduchon metrics, (that is, $\partial\bar\partial\omega^{n-1} = 0$, equivalently, $d^*_{\omega}\vartheta = 0$,) one has
$$\int_X \Delta^{Ch}_{\omega} f \, d\mu_{\omega} = 0$$
- for balanced metrics in the sense of Michelsohn, (that is, $d\omega^{n-1} = 0$, equivalently, $\vartheta = 0$,)
$$\Delta^{Ch}_{\omega} = \Delta_{d,\omega} \text{ on } \mathcal{C}^{\infty}(X;\mathbb{C}).$$

In order to determine, a priori, the value of the constant Chern-scalar curvature, we fix a normalization on the expected solution.

Let:

$$\{\omega\}_1 := \left\{ \exp(2f/n)\,\eta \in \{\omega\} : \int \exp(2f/n)\, d\mu_{\eta} = 1 \right\} \subseteq \{\omega\},$$

(where η, again, is the unique Gauduchon metric in $\{\omega\}$ with volume 1).

SCUOLA NORMALE SUPERIORE PISA

In this way, in the equation

$$[CY] \qquad \Delta^{Ch}_{\eta} f + S^{Ch}(\eta) = \lambda \exp(2f/n)$$

we have

$$\lambda = \int S^{Ch}(\eta) = \frac{1}{(n-1)!} \int_X c_1^{BC}(K_X^{-1}) \wedge \eta^{n-1},$$

which is the Gauduchon degree of the conformal class, denoted $\Gamma_X(\{\omega\})$.
~~By the work~~
It corresponds to the degree of the anti-canonical line bundle K_X^{-1}. By the work by GAUDUCHON, the degree of a holomorphic line-bdl is equal to the volume of the divisor associated to any meromorphic section by means of the Gauduchon metric.

Whence we get the following.

PROP. Let X be cpt cplx. ~~If Kod(X) >~~
~~1~~ If $\mathrm{Kod}(X) \geq 0$, then $\Gamma_X(\{\omega\}) \leq 0$, $\forall \{\omega\}$.
~~If Kod(X) = 0 then~~

This is exactly the "easy case" for solving (CY) equation.

RMK Note that Inoue surfaces have $\mathrm{Kod} = -\infty$, and $\Gamma_X(\{\omega\}) < 0$ for any $\{\omega\}$ by TELEMAN.

§ SOLUTION OF CHERN-YAMABE EQ. FOR ZERO GAUDUCHON DEGREE

In case of zero Gauduchon degree, (CY) equation reduces to a linear elliptic equation:

THM Let X be cpt cplx, let $\{\omega\}$ conf. class of Herm metrics. If $\Gamma_X(\{\omega\})=0$, then
$$\mathcal{CY}(X;\{\omega\}) = \{pt\}$$
(in fact, ~~up~~ even before quotienting by $\mathcal{H}\mathrm{Conf}(X;\{\omega\})$).
That is, there exists a unique $\omega_p \in \{\omega\}_1$ with constant Chern scalar curvature, equal to ~~zero~~.
Moreover, $\mathcal{H}\mathrm{Conf}(X;\{\omega\}) \simeq \mathcal{H}\,\mathrm{Isom}(X;\omega_p)$.

§ SOLUTION OF CHERN-YAMABE EQ. FOR NEGATIVE GAUDUCHON DEGREE

The case of negative Gauduchon degree is a bit more interesting from the analytic point of view, but still easy:

THM Let X be cpt cplx, let $\{\omega\}$ conf. class of Herm. metr. If $\Gamma_X(\{\omega\})<0$, then
$$\mathcal{CY}(X;\{\omega\}) = \{pt\}$$
(in fact, even before quotienting by $\mathcal{H}\mathrm{Conf}(X;\{\omega\})$.)
That is, there exists a unique $\omega_p \in \{\omega\}_1$ with constant Chern scalar curvature, which ~~being~~ is negative.
Moreover, $\mathcal{H}\mathrm{Conf}(X;\{\omega\}) \simeq \mathcal{H}\mathrm{Isom}(X;\omega_p)$.

Sketch of proof

We're assuming
$$\Gamma_X(\{\omega\}) = \int_X S^{ch}(\eta)\, d\mu_\eta < 0 .$$

In fact, by elliptic theory, we may choose $\omega \in \{\omega\}$ such that, at every point,
$$S^{ch}(\omega) < 0 . \quad \text{~~at every point~~}$$
We fix such an ω as reference metric.

We use continuity method with path
$$[CY_t] \qquad \Delta^{ch}_{\omega} f + t\, S^{ch}(\omega) = \lambda \exp(2f/n) - \lambda(1-t) .$$

- Clearly, $f=0$ is a solution for $(CY_{t=0})$.
- The openness follows by noting that the linearized operator,
$$v \mapsto Dv := \Delta^{ch}_{\omega} v - \lambda \exp(2f_0/n) \cdot 2v/n ,$$
is elliptic and with index 0. So it is bijective.
- The closedness follows by uniform L^∞-estimates:

LEMMA Let $\{f_{t_n}\}_n \subseteq C^{2,\alpha}(X;\mathbb{R})$, where f_{t_n} is a solution of (CY_{t_n}) with $\{t_n\}_n \subseteq [0,1]$. Then there exits $C_0 = C_0(X,\omega,\lambda) > 0$ s.t. $\forall n$,
$$\| f_{t_n} \|_{L^\infty} \leq C_0 .$$

The uniqueness follows from maximum principle.

The ~~t-conformal~~ canonicalness follows by uniqueness. $\square$

§ TOWARDS CHERN-YAMABE PROBLEM FOR POSITIVE GAUDUCHON DEGREE

There are examples of manifolds with $\Gamma_X(\{\omega\})>0$ and constant Chern scalar curvature metrics in the chosen conformal class $\{\omega\}$.

example. The Hopf surface $\mathbb{C}^2\setminus\{0\}/_{\langle z\mapsto z/2\rangle}$ has Kod $=-\infty$, and "antiKod ≥ 0", so, $\forall\{\omega\}$, it holds $\Gamma_X(\{\omega\})>0$.

The standar Herm metric $\frac{1}{|z|^2}\omega_0$, where ω_0 is the flat metric on $\mathbb{C}^2$, has constant Chern scalar curvature.

~~By~~ Implicit Function Thm, Chern-Yamabe problem ~~should~~ may serve in positive case ~~may have solution~~, at least when $S^{Ch}(\eta)$ is close to zero (which yields $\Gamma_X(\{\omega\})$ positive and small)

An ~~Ex~~ example in the same spirit:

PROP. Let X cpt cplx mfd, $\dim_{\mathbb{C}} X\geq 2$, let $\{\omega\}$ conformal class of Herm. metrics with $\Gamma_X(\{\omega\})>0$. Then, for any curve Σ_g of genus g, the mfd $X\times\Sigma_g$ admits metrics with positive constant Chern scalar curvature.

Chern-Yamabe problem has a variational formulation in case $\eta\in\{\omega\}$ is balanced (ie, $\theta_\eta=0$):

PROP. Let X be cpt cplx, let $\{\omega\}$ conformal class of Herm. metrics with $\eta\in\{\omega\}$ balanced in the sense of Michelsohn.

Then (CY) is the Euler-Lagrange equation of

$$\mathcal{F}(f):=\frac{1}{2}\int_X |df|^2_\eta \, d\mu_\eta+\int_X S^{Ch}(\eta)\, f\, d\mu_\eta$$

with constraint

$$\int_X \exp(2f/n)\, d\mu_\eta=1$$