

"SPECIAL METRICS ON COMPLEX MANIFOLDS:
ON THE CHERN-YAMABE PROBLEM"

Work in progress of / SIMONE CALAMAI
CRISTIANO SPOTTI
arXiv: 1501.02638 [math.DG]

§ INTRODUCTION

AIM. Search for special (possibly "canonical" - meaning the natural transf. preserves the metric) [Hermitian] metrics on cptplx mfd's.

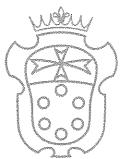
What "special metric" means:

- with curvature properties:
(eg.: constant scalar curvature; ~~Kähler~~ Einstein; extremal; ...)
- with "cohomological" properties generalizing Kähler condition:
(eg.: Gauduchon; balanced in the sense of Michelsohn; pluriclosed or SKT; ...)
- a combination of the two:
(eg.: Kähler-Einstein; Koll; ...)

Two results for this aim:

THEM (GAUDUCHON). ~~Any conformal class~~ Any Hermitian-conformal class on a cptplx n -manifold contains a standard metric satisfying $\partial\bar{\partial}\omega^{n-1} = 0$, unique up to homothety.

It is called Gauduchon metric.



SCUOLA
NORMALE
SUPERIORE
PISA

THEM (YAMABE; TRUBINGER; AUBIN; SCHOEN). Any conformal class on a cpt differentiable manifold contains a constant scalar curvature metric.

§ CHERN-YAMABE PROBLEM

We investigate an analogue of the Yamabe problem for cplx manifolds and Herm-conformal structures.

Since, for non-Kähler metrics g , ∇^{LC} does not preserve the cplx structure \mathcal{J} , we ~~we~~ consider Hermitian connections, that is, connections preserving both the metric and the cplx structure, and ~~then~~ having torsion.

One restricts to canonical connections in the sense of Gauduchon: they are a 1-parameter family with prescribed torsion:

$$T_{\mathcal{B}_t}^{1,1} = 0 \quad (\text{where: } \mathcal{B}_t := \mathcal{B} - t\mathcal{B}_c, \\ \mathcal{B}_c := \frac{1}{2}g(\mathcal{B}_c, \dots, -\mathcal{J})^\sharp).$$

In particular, we consider the Chern connection, corresponding to $t=1$:

it is the unique Herm. connection with $T^{1,1}=0$,

equivalently, the unique connection on $T^{1,0}X$ preserving the Herm structure and whose $(0,1)$ -part coincides with the Cauchy-Riemann operator associated to the holomorphic structure.

Its scalar curvature is given by

$$S^{Ch}(w) = \text{tr}_w i \bar{\partial} \partial \log w^n.$$



SCUOLA
NORMALE
SUPERIORE
PISA

⚡ (Similar argument may possibly work for other canonical connections; we consider Chern conn for the rôle of Chern-Picci curvature in non-Kähler Calabi-Yau, after the work by, eg, Tosatti and Weinkove.)

More precisely:

Let X be spt cplx mfd, $\dim_{\mathbb{C}} X = n$.

Fix $\{\omega\}$ conformal class of Hermitian metrics.

We denote by $\eta \in \{\omega\}$ the unique Goursat metric in $\{\omega\}$ with $\int_X d\mu_{\eta} = 1$.

We have a natural action:

$$G_X(\{\omega\}) := \text{BilConf}(X; \{\omega\}) \times \mathbb{R}^+ \curvearrowright \{\omega\},$$

where $\text{BilConf}(X; \{\omega\})$ denotes the space of biholom. automorphisms of X preserving $\{\omega\}$.

We study the moduli space:

$$\mathcal{EY}(X; \{\omega\}) := \left\{ \omega' \in \{\omega\} : \text{Sch}(\omega') \text{ const.} \right\} / G_X(\{\omega\})$$

As a first problem, we have the following analogue to the Yamabe problem:

CHERN-YAMABE PROBLEM. Let X be spt cplx, let $\{\omega\}$ be a conformal class of Hermitian metrics. Prove $\mathcal{EY}(X; \{\omega\}) \neq \emptyset$.

RMK. By LIU-YANG [arxiv:1404.2481], if the Chern scalar curvature is equal ~~to~~ to the scalar curvature (resp., to the S -scalar curvature) of the corresponding Riemannian metric, then the metric is stable.

Then, Chern-Yamabe problem goes in different direction wrt both Yamabe pb, and Yamabe pb for altn-Hermitian mfd as studied by del Rio and Simanca.



We ~~give~~ translate the Chern-Yamabe problem in ~~an~~ ~~and~~ a semilinear PDE of Liouville-type.

~~By changing~~ Under conformal transformations, the Chern-scalar curvature changes as:

$$S^{ch}(\exp(2f/m)\omega) = \exp(-2f/m) (S^{ch}(\omega) + \Delta_{\omega}^{ch} f).$$

Here, $\Delta_{\omega}^{ch} f := (\omega, dd^c f)_{\omega} = 2i \int \omega \bar{\partial} \partial f$ denotes the Laplacian associated to the Chern connection.

In terms of the de Rham Laplacian ($\Delta_{d,\omega} = d_{\omega}^{*} d$), one writes:

$$\Delta_{\omega}^{ch} f = \Delta_{d,\omega} f + (df | \bar{\sigma})_{\omega},$$

where $\bar{\sigma}$ denotes the Lee-form of ω (that is, $d\omega^{n-1} = \bar{\sigma} \wedge \omega^{n-1}$).

In particular, note that:

- for Gauduchon metrics, (that is, $\bar{\partial} \omega^{n-1} = 0$, ~~or~~ equivalently, $d_{\omega}^{*} \bar{\sigma} = 0$), one has

$$\int_X \Delta_{\omega}^{ch} f \, d\mu_{\omega} = 0$$

- for balanced metrics in the sense of Michelsohn, (that is, $d\omega^{n-1} = 0$, equivalently, $\bar{\sigma} = 0$),

$$\Delta_{\omega}^{ch} = \Delta_{d,\omega} \text{ on } C^{\infty}(X; \mathbb{R}).$$

In order to determine, a priori, the value of the constant Chern-scalar curvature, we fix a normalization on the expected solution.

Let:

$$\{\omega\}_1 := \left\{ \exp(2f/m) \eta \in \{\omega\} : \int \exp(2f/m) \, d\mu_{\eta} = 1 \right\} \\ \subseteq \{\omega\},$$

(where η , again, is the unique Gauduchon metric in $\{\omega\}$ with volume 1).



In this way, in the equation

$$(CY) \quad \Delta_{\eta}^{ch} f + S^{ch}(\eta) = \lambda \exp(2f/m)$$

we have

$$\lambda = \int S^{ch}(\eta) = \frac{1}{(n-1)!} \int_X e_1^{BC}(K_X^{-1}) \eta^{n-1},$$

which is the Gauduchon degree of the conformal class, denoted $\Gamma_X(\{\omega\})$.

~~By the work~~

It corresponds to the degree of the anti-canonical line bundle K_X^{-1} . By the work by GAUDUCHON, the degree of a holomorphic line-bundle is equal to the volume of the divisor associated to any meromorphic section by means of the Gauduchon metric.

Whence we get the following:

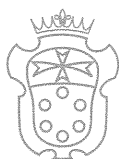
PROP. Let X be a cplx. ~~If $Kod(X) > 0$~~

~~If $Kod(X) > 0$~~ If $Kod(X) > 0$, then $\Gamma_X(\{\omega\}) \leq 0$, $\forall \{\omega\}$.

~~If $Kod(X) > 0$~~

This is exactly the "easy case" for solving (CY) equation.

RMK Note that gauge surfaces have $Kod = -\infty$, and $\Gamma_X(\{\omega\}) < 0$ for any $\{\omega\}$ by TEJEMAN.



§ SOLUTION OF CHERN-YAMABE EQ. FOR ZERO GAUßBUCHON DEGREE

In case of zero Gauss-Bonnet degree, (CY) equation reduces to a linear elliptic equation:

THM Let X be cpt sph, let $\{\omega\}$ conf. class of Riemann metrics.

If $\Gamma_X(\{\omega\}) = 0$, then

$$CY(X; \{\omega\}) = \{c\}$$

(in fact, ~~up~~ even before quotienting by $\mathcal{H}Conf(X; \{\omega\})$).

That is, there exists a unique $\omega_c \in \{\omega\}$, with constant Chern scalar curvature, equal to c .

Moreover, $\mathcal{H}Conf(X; \{\omega\}) \simeq \mathcal{H}Isom(X; \omega_c)$.

§ SOLUTION OF CHERN-YAMABE EQ. FOR NEGATIVE GAUßBUCHON DEGREE

The case of negative Gauss-Bonnet degree is a bit more interesting from the analytic point of view, but still easy:

THM Let X be cpt sph, let $\{\omega\}$ conf. class of Riemann metr.

If $\Gamma_X(\{\omega\}) < 0$, then

$$CY(X; \{\omega\}) = \{c\}$$

(in fact, even before quotienting by $\mathcal{H}Conf(X; \{\omega\})$.)

That is, there exists a unique $\omega_c \in \{\omega\}$, with constant Chern scalar curvature, which ~~being~~ is negative.

Moreover, $\mathcal{H}Conf(X; \{\omega\}) \simeq \mathcal{H}Isom(X; \omega_c)$.



Sketch of proof

We're assuming

$$\Gamma_X(\{\omega\}) = \int_X S^{ch}(m) d\mu_m < 0.$$

In fact, by elliptic theory, we may choose $\omega \in \{\omega\}$ such that, at every point,

$$S^{ch}(\omega) < 0. \quad \text{strong point}$$

We fix such an ω as reference metric.

We use continuity method with path

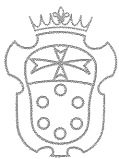
$$[CY_X] \quad \Delta_{\omega}^{ch} f + t S^{ch}(\omega) = \lambda \exp(2f/m) - \lambda(1-t).$$

- Clearly, $f=0$ is a solution for $(CY_{X=0})$.
- The openness follows by noting that the linearized operator,
$$v \mapsto Dv := \Delta_{\omega}^{ch} v - \lambda \exp(2f_0/m) \cdot \frac{2v}{m},$$
is elliptic and with index 0. So it is bijective.
- The closedness follows by uniform L^{∞} -estimates:

LEMMA Let $\{f_{t_n}\}_m \in C^{2,\alpha}(X; \mathbb{R})$, where f_{t_n} is a solution of $(CY_{X_{t_n}})$ with $\{t_n\}_m \subseteq [0, 1]$. Then there exists $c_0 = c_0(X, \omega, \lambda) > 0$ s.t., $\forall n$,
$$\|f_{t_n}\|_{L^{\infty}} \leq c_0.$$

The uniqueness follows from maximum principle.

The ~~strong~~ canonicalness follows by uniqueness. \square



§ TOWARDS CHERN-YAMABE PROBLEM FOR POSITIVE GAUDDUCHON DEGREE

There are examples of manifolds with $\Gamma_X(\{\omega\}) > 0$ and constant Chern scalar curvature metrics in the chosen conformal class $\{\omega\}$.

example. The Riemann surface $\mathbb{C}^2 \setminus \{0\} / \langle \mathbb{Z} \mapsto z/2 \rangle$ has $\text{Kod} = -\infty$, and "antikod > 0 ", $\forall \omega$, it holds $\Gamma_X(\{\omega\}) > 0$.

The standard Kähler metric $\frac{1}{|z|^2} \omega_0$, where ω_0 is the flat metric on \mathbb{C}^2 , has constant Chern scalar curvature.

~~By~~ Implicit Function Thm, Chern-Yamabe problem ~~should~~ ~~may~~ ~~never~~ ~~in~~ ~~positive~~ ~~case~~ ~~at~~ ~~least~~ when $S^{ch}(\eta)$ is close to zero (which yields $\Gamma_X(\{\omega\})$ positive and small).

On ~~the~~ example in the same spirit:

PROP. Let X cpt cplx mfd, $\dim_{\mathbb{C}} X \geq 2$, let $\{\omega\}$ conformal class of Kähler metrics with $\Gamma_X(\{\omega\}) > 0$. Then, for any curve Σ_g of genus g , the mfd $X \times \Sigma_g$ admits metrics with positive constant Chern scalar curvature.

Chern-Yamabe problem has a variational formulation in case $\eta \in \{\omega\}$ is balanced (i.e., $\Theta_\eta = 0$):

PROP. Let X be cpt cplx, let $\{\omega\}$ conformal class of Kähler metrics with $\eta \in \{\omega\}$ balanced in the sense of Michelsohn.

Then (c_Y) is the Euler-Lagrange equation of $\mathcal{F}(f) := \frac{1}{2} \int_X |df|_n^2 d\mu_\eta + \int_X S^{ch}(\eta) f d\mu_\eta$

with constraint

$$\int_X \exp(2f/n) d\mu_\eta = 1$$

