THE $r^\sharp$ INVARIANT AS A DISCRIMINANT FOR THE SURVIVAL OF THE H-FLUX UNDER T-DUALITY ON PRODUCT MANIFOLDS

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Abstract. We show that the cohomological invariant $r^\sharp$, introduced in [1] as a lower bound for the off-diagonal holonomy dimension of metric connections with totally skew torsion on product manifolds, predicts the behaviour of the torsion 3-form under both dimensional reduction and Buscher T-duality. On a product $M = \Sigma_g \times M_2$ equipped with a product metric, when $r^\sharp = 0$ the parallel-form strata identify a flat circle factor $S^1_\beta \subset M_2$ via the de Rham splitting theorem, and the entire H-flux is converted into geometric flux under T-duality along S^1_β (the parallel regime); when $r^\sharp = 1$, no such circle factor exists, and the H-flux survives T-duality along every flat circle factor as H-flux in the dual background (the transversely irreducible regime). When $M_2 = N \times T^k$ contains a torus factor, we prove that the Bouwknegt–Evslin–Mathai obstruction to successive T-dualities vanishes automatically for H-flux of pure bidegree $(2, 1)$, that the resulting dualities are non-interfering and order-independent, and that $r^\sharp$ detects the *irreducible kernel* of the H-flux: the component that survives T-duality along every flat circle factor and cannot be converted into geometric or non-geometric flux in any duality frame. This provides a metric refinement of topological T-duality: while the latter disregards the Riemannian metric entirely, $r^\sharp$ detects whether the cohomological coupling is aligned with the flat sub-factors identified by the Levi-Civita parallel-form strata.

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1. Introduction

Let (M, g) be a compact oriented Riemannian product manifold and let $\nabla^C = \nabla^{LC} + \frac{1}{2}T$ be a metric connection with totally skew-symmetric torsion $T \in \Omega^3(M)$. In [1], the authors introduced the invariant

$$r^\sharp \;:=\; \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) \;-\; \dim \mathcal{K},$$

where the torsion 3-form represents a non-trivial de Rham mixed cohomology class $[\omega]$ via its harmonic projection ($T^\flat = \omega_h + d\alpha + \delta\beta$, i.e. $[\omega] := [\omega_h]$) and $\mathcal{K}$ is an obstruction space measuring the intersection of the mixed Künneth factor spaces with the Levi-Civita parallel-form strata of the factors. The main theorem of [1] and its extension (in *Section 7* of [1]) establish that, on product manifolds where the factor contributing the 2-form is a compact oriented surface while the other factor may be of arbitrary dimension, $r^\sharp$ provides a lower bound for the dimension of the off-diagonal holonomy algebra $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C)$ on a non-empty open subset.

As observed in [1], the proof for pure bidegree $(2,1)$ uses $\dim M_1 = 2$ but not $\dim M_2 = 2$; the theorem therefore extends to products $\Sigma_g \times M_2$ where Σ_g is a compact oriented surface and M_2 is a compact oriented Riemannian manifold of arbitrary dimension. In the present note we exploit this extended setting to establish a connection between $r^\sharp$ and two fundamental operations: dimensional reduction along a circle factor, and Buscher T-duality.

Throughout, Σ_g denotes a compact oriented surface of genus $g \geq 1$ and (M_2, g_2) a compact oriented Riemannian manifold with $b_1(M_2) \geq 1$. We set $M := \Sigma_g \times M_2$ with the product metric $g = g_\Sigma \oplus g_2$ and consider torsion 3-forms of pure bidegree $(2,1)$ with respect to the splitting $T_pM = V_\Sigma \oplus V_{M_2}$, i.e. $T \in \Gamma(\Lambda^2 V_\Sigma^* \otimes V_{M_2}^*)$. Since $\dim \Sigma_g = 2$, every such form can be written pointwise as

$$T_p \;=\; h(p)\, \mathrm{vol}_{V_\Sigma} \wedge \tau_p, \tag{1.1}$$

where $h\colon M \to \mathbb{R}$ is smooth and $\tau \in \Gamma(V_{M_2}^*)$.

When $r^\sharp = 0$, the harmonic 1-form $\beta \in H^1(M_2)$ that enters the mixed class belongs to $\mathcal{P}_1(M_2)$, defining a parallel—hence Killing—vector field $\beta^\sharp$ on M_2. When the orbits of $\beta^\sharp$ are closed, the de Rham splitting theorem yields $M_2 \cong N \times S^1_\beta$, where S^1_β is the flat circle generated by $\beta^\sharp$ and N is the complementary factor. This circle is the natural candidate for dimensional reduction and T-duality.

In the remainder of the paper we work in the concrete setting where M_2 admits an explicit circle factor, i.e. $M_2 = N \times S^1$ for a compact oriented Riemannian manifold (N, g_N) with $b_1(N) \geq 1$. This is the setting in which both dimensional reduction and Buscher T-duality can be formulated directly. The general mechanism described above—whereby $r^\sharp = 0$ produces a circle factor via the de Rham splitting theorem—is then realised concretely: S^1_β coincides with S^1 when $\beta \in \mathbb{R} \cdot d\theta$, and with a different circle factor in the de Rham splitting of $N \times S^1$ when $\beta \in \mathcal{P}_1(N)$.

2. Dimensional reduction and the r-sharp invariant

We begin by recalling the relevant definitions from [1]. The harmonic projection of $T^\flat$ with respect to g decomposes according to the Künneth theorem as $\omega_h^{2,1} = \mathrm{vol}_{\Sigma_g} \wedge \beta$ with $\beta \in \mathcal{H}^1(N \times S^1, g_{N \times S^1})$. Since $b_2(\Sigma_g) = 1$, the mixed factor space $\mathcal{V}_{2,1}$ is intrinsically defined [1, Lemma 2.3]. The obstruction space is $\mathcal{K} = \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(\Sigma_g) \otimes \mathcal{P}_1(N \times S^1)\big)$, and the invariant is $r^\sharp = r_{2,1} - \dim \mathcal{K}$, where $r_{2,1} \in \{0, 1\}$ in the extended admissible setting.

The parallel-form stratum of the second factor decomposes as

$$\mathcal{P}_1(N \times S^1) \;=\; \mathcal{P}_1(N) \;\oplus\; \mathbb{R} \cdot d\theta. \tag{2.1}$$

Consequently, $\beta \in \mathcal{P}_1(N \times S^1)$ if and only if β can be written as a sum of a parallel 1-form on N and a multiple of $d\theta$. In particular, when $\mathcal{P}_1(N) = \{0\}$ (e.g. when $\mathrm{Ric}(N) < 0$, since a parallel vector field X satisfies $\mathrm{Ric}(X, X) = 0$, contradicting strict negativity unless $X = 0$), the condition $\beta \in \mathcal{P}_1(N \times S^1)$ reduces to $\beta \in \mathbb{R} \cdot d\theta$.

We now examine the behaviour of the torsion under the natural restriction to the reduced product $\Sigma_g \times N$, obtained by collapsing the circle factor.

Proposition 2.1 (Dimensional reduction along S^1)**.** *Let $M = \Sigma_g \times N \times S^1$ with product metric and let $T = \mathrm{vol}_{\Sigma_g} \wedge \beta$ be a harmonic torsion 3-form of pure bidegree $(2,1)$ with $\beta \in \mathcal{H}^1(N \times S^1)$. Denote by $\iota\colon \Sigma_g \times N \hookrightarrow M$ the inclusion at any fixed value of θ.*

- (i) *If $\beta \in \mathcal{H}^1(N)$ (i.e. β has no component along $d\theta$), then $\iota^*T = \mathrm{vol}_{\Sigma_g} \wedge \beta \neq 0$ as a 3-form on $\Sigma_g \times N$.*
- (ii) *If $\beta = c\, d\theta$ for some $c \in \mathbb{R} \setminus \{0\}$, then $\iota^*T = 0$.*

Proof. The pullback ι^* acts on 1-forms on $N \times S^1$ by restriction to vectors tangent to N (at a fixed value of θ). For any vector Z tangent to N, $\iota^*(d\theta)(Z) = d\theta(Z) = 0$, since Z has no component along ∂_θ. Thus $\iota^*(d\theta) = 0$. Conversely, if $\beta \in \Omega^1(N)$, then for any Z tangent to N, $\iota^*\beta(Z) = \beta(Z)$, and $\iota^*\beta = \beta$ as a 1-form on N.

In case (i), $T = \mathrm{vol}_{\Sigma_g} \wedge \beta$ with $\beta \in \Omega^1(N)$, so $\iota^*T = \mathrm{vol}_{\Sigma_g} \wedge \iota^*\beta = \mathrm{vol}_{\Sigma_g} \wedge \beta$. Since β is a non-zero harmonic 1-form on N and vol_{Σ_g} is non-zero, the wedge product is a non-zero 3-form on $\Sigma_g \times N$.

In case (ii), $T = c\, \mathrm{vol}_{\Sigma_g} \wedge d\theta$, so $\iota^*T = c\, \mathrm{vol}_{\Sigma_g} \wedge \iota^*(d\theta) = 0$. □

When $\mathcal{P}_1(N) = \{0\}$, the invariant $r^\sharp$ determines exactly which of the two cases occurs.

Corollary 2.2 (r-sharp and survival of irreducibility)**.** *Under the hypotheses of Proposition 2.1, assume further that $\mathcal{P}_1(N) = \{0\}$. Then:*

- (i) *If $r^\sharp = 1$, the torsion 3-form survives restriction to $\Sigma_g \times N$, and the restricted connection is irreducible on the reduced product (by* [1, Corollary 5.3] *applied to $\Sigma_g \times N$).*
- (ii) *If $r^\sharp = 0$ with $r_{2,1} = 1$, then $\beta = c\, d\theta$, the torsion vanishes on $\Sigma_g \times N$, and the restricted connection reduces to the Levi-Civita connection of the product, which has reducible holonomy.*

Proof. Since $\mathcal{P}_1(N) = \{0\}$, the decomposition (2.1) gives $\mathcal{P}_1(N \times S^1) = \mathbb{R} \cdot d\theta$. The condition $r^\sharp = 0$ with $r_{2,1} = 1$ means $\beta \in \mathcal{P}_1(N \times S^1) = \mathbb{R} \cdot d\theta$, so $\beta = c\, d\theta$ for some $c \neq 0$. Proposition 2.1(ii) gives $\iota^*T = 0$. The restricted connection on $\Sigma_g \times N$ is therefore $\nabla^{LC}_{\Sigma_g \times N}$, whose holonomy preserves the splitting $V_\Sigma \oplus V_N$ and is thus reducible.

When $r^\sharp = 1$, we have $\beta \notin \mathcal{P}_1(N \times S^1)$. Since $\mathcal{P}_1(N \times S^1) = \mathbb{R} \cdot d\theta$, writing $\beta = \gamma + c\, d\theta$ with $\gamma \in \mathcal{H}^1(N)$ and $c \in \mathbb{R}$, the condition $\beta \notin \mathbb{R} \cdot d\theta$ forces $\gamma \neq 0$. Proposition 2.1(i) applied to the component γ gives $\iota^*T = \mathrm{vol}_{\Sigma_g} \wedge \gamma \neq 0$ on $\Sigma_g \times N$. This is a non-trivial torsion 3-form of pure bidegree $(2,1)$ on the product $\Sigma_g \times N$, where Σ_g is a surface. Since $\gamma \in \mathcal{H}^1(N) \setminus \{0\}$ and $\mathcal{P}_1(N) = \{0\}$, the restricted torsion class $[\iota^* T]$ is a non-trivial mixed class on $\Sigma_g \times N$ with $r_{2,1} = 1$ and $\dim \mathcal{K} = 0$. By [1, Corollary 5.3], the holonomy of the restricted connection is irreducible on $\Sigma_g \times N$. □

Remark 2.3. When $\mathcal{P}_1(N) \neq \{0\}$, the invariant $r^\sharp$ still determines the existence of a *specific* circle factor along which the torsion vanishes: the parallel 1-form $\beta \in \mathcal{P}_1(N \times S^1)$ defines a parallel vector field $\beta^\sharp$, which is automatically Killing ($\nabla \beta^\sharp = 0$ implies $\mathcal{L}_{\beta^\sharp} g = 0$). When the orbits of $\beta^\sharp$ are closed—which is the case whenever β is tangent to an explicit circle factor in the de Rham splitting of $N \times S^1$—they generate a flat circle S^1_β. Dimensional reduction along S^1_β annihilates the torsion, while reduction along any other circle factor generically does not.

When $r^\sharp = 1$, no parallel 1-form is associated to β, hence no circle factor exists along which the torsion can be annihilated. The irreducibility is resistant to dimensional reduction along any flat circle factor of $N \times S^1$.

3. T-duality and the r-sharp invariant

We now connect the dimensional reduction picture to the Buscher rules for T-duality. We work on $M = \Sigma_g \times N \times S^1$ with product metric $g = g_\Sigma \oplus g_N \oplus d\theta^2$. The vector field ∂_θ is a Killing field, and the Buscher rules [3, 4] for T-duality along S^1 apply.

In the notation of Buscher, the background fields are the metric g_{MN}, the B-field B_{MN} (with $H = dB$ the H-flux), and the dilaton ϕ. On a product metric with $g_{\theta\mu} = 0$ and $g_{\theta\theta} = 1$, the Buscher rules simplify considerably.

Proposition 3.1 (Buscher rules on a product metric)**.** *Let $M = \Sigma_g \times N \times S^1$ with product metric $g = g_\Sigma \oplus g_N \oplus d\theta^2$ and H-flux $H = \mathrm{vol}_{\Sigma_g} \wedge \beta$ with $\beta \in \mathcal{H}^1(N \times S^1)$. Write $\beta = \gamma + c\, d\theta$ with $\gamma \in \mathcal{H}^1(N)$ and $c \in \mathbb{R}$. Under Buscher T-duality along S^1:*

(i) *The component $H^\parallel := c\, \mathrm{vol}_{\Sigma_g} \wedge d\theta$ (with one index along θ) is mapped to off-diagonal components of the dual metric: $\tilde{g}_{\theta\mu}$. It becomes geometric flux in the T-dual background and ceases to exist as H-flux.*

(ii) *The component $H^\perp := \mathrm{vol}_{\Sigma_g} \wedge \gamma$ (with no index along θ) is invariant: $\widetilde{H}^\perp = H^\perp$. It remains H-flux in the T-dual background.*

Proof. In coordinates (x^1, x^2) on Σ_g, (y^a) on N, and θ on S^1, the product metric satisfies $G_{\theta\theta} = 1$ and $G_{\mu\theta} = 0$ for all $\mu \neq \theta$. The B-field satisfies $H = dB$; we decompose B into components $B_{\mu\theta}$ and $B_{\mu\nu}$, where μ, ν range over Σ_g and N directions only. Since $\partial_\theta B_{\mu\nu} = 0$ on the product, the components of H satisfy $H_{12\theta} = \partial_1 B_{2\theta} - \partial_2 B_{1\theta}$ and $H_{12a} = \partial_1 B_{2a} - \partial_2 B_{1a} + \partial_a B_{12}$.

Writing $\beta = \gamma + c\, d\theta$ with $\gamma \in \mathcal{H}^1(N)$ and $c \in \mathbb{R}$, the component along $d\theta$ gives $H_{12\theta} = c\sqrt{g_\Sigma}$, and the component along γ gives $H_{12a} = \sqrt{g_\Sigma}\, \gamma_a$. When $c \neq 0$ we have $H_{12\theta} \neq 0$, which forces $B_{\mu\theta} \neq 0$ for at least some μ.

The Buscher rules [3, 4] for T-duality along S^1, applied to a background with $G_{\theta\theta} = 1$ and $G_{\mu\theta} = 0$, give the dual fields

$$\tilde{G}_{\theta\theta} = 1, \qquad \tilde{G}_{\mu\theta} = B_{\mu\theta}, \qquad \tilde{G}_{\mu\nu} = G_{\mu\nu} + B_{\mu\theta}B_{\nu\theta}, \tag{3.1}$$

$$\tilde{B}_{\mu\theta} = 0, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu}. \tag{3.2}$$

For the dual H-flux $\tilde{H} = d\tilde{B}$, the components with a θ-index satisfy $\tilde{H}_{\mu\nu\theta} = \partial_\mu \tilde{B}_{\nu\theta} - \partial_\nu \tilde{B}_{\mu\theta} + \partial_\theta \tilde{B}_{\mu\nu} = 0$, since $\tilde{B}_{\mu\theta} = 0$ by (3.2) and $\partial_\theta \tilde{B}_{\mu\nu} = \partial_\theta B_{\mu\nu} = 0$ on the product. Thus all H-flux components with a θ-index vanish in the dual background; the information formerly carried by $H_{12\theta}$ is now encoded in the off-diagonal metric components $\tilde{G}_{\mu\theta} = B_{\mu\theta}$ by (3.1). This is the geometric flux.

For the components without θ-index: $\tilde{H}_{12a} = \partial_1 \tilde{B}_{2a} - \partial_2 \tilde{B}_{1a} + \partial_a \tilde{B}_{12} = \partial_1 B_{2a} - \partial_2 B_{1a} + \partial_a B_{12} = H_{12a}$. These components are unchanged: the H-flux along γ survives as H-flux in the dual background. □

The connection with $r^\sharp$ is now immediate.

Theorem 3.2 (r-sharp predicts T-duality behaviour)**.** *Let $M = \Sigma_g \times N \times S^1$ with product metric and torsion $T = \mathrm{vol}_{\Sigma_g} \wedge \beta$ of pure bidegree $(2, 1)$. Assume $\mathcal{P}_1(N) = \{0\}$.*

- (i) *If $r^\sharp = 1$, H-flux survives T-duality along S^1 as H-flux in the dual background. The irreducibility of the connection is preserved under both dimensional reduction and T-duality.*
- (ii) *If $r^\sharp = 0$ with $r_{2,1} = 1$, the entire H-flux is converted into geometric flux under T-duality along S^1. The torsion vanishes on the dimensionally reduced space $\Sigma_g \times N$, and the reduced connection has reducible holonomy.*

In the general case (without the hypothesis $\mathcal{P}_1(N) = \{0\}$), $r^\sharp = 1$ guarantees that the H-flux is resistant to T-duality along any flat circle factor of $N \times S^1$; when $r^\sharp = 0$, the parallel 1-form β identifies a Killing vector field $\beta^\sharp$ on $N \times S^1$. If the orbits of $\beta^\sharp$ are closed (which is automatic when β is tangent to an explicit circle factor in the de Rham splitting, as in all examples of Section 4), T-duality along the corresponding circle S^1_β converts the entire H-flux into geometric flux.

Proof. When $\mathcal{P}_1(N) = \{0\}$, the condition $r^\sharp = 0$ with $r_{2,1} = 1$ forces $\beta = c\,d\theta$ (as in the proof of Corollary 2.2). Proposition 3.1(i) then shows that the entire H-flux becomes geometric flux. Corollary 2.2(ii) shows that the torsion vanishes on $\Sigma_g \times N$.

When $r^\sharp = 1$, β has a non-zero component $\gamma \in \mathcal{H}^1(N) \setminus \{0\}$. Proposition 3.1(ii) shows that this component survives as H-flux. Corollary 2.2(i) shows that the restricted connection on $\Sigma_g \times N$ is irreducible.

For the general case, the argument of Remark 2.3 applies: $r^\sharp = 0$ means $\beta \in \mathcal{P}_1(N \times S^1)$, so the vector field $\beta^\sharp$ is parallel and hence Killing ($\nabla \beta^\sharp = 0$ implies $\mathcal{L}_{\beta^\sharp} g = 0$). When the orbits of $\beta^\sharp$ are closed, they generate a circle factor S^1_β in the de Rham splitting of $N\times S^1$, and T-duality along S^1_β converts the H-flux entirely into geometric flux by the same Buscher calculation (applied with S^1_β in place of S^1). When $r^\sharp = 1$, $\beta \notin \mathcal{P}_1(N \times S^1)$, so β is not aligned with any flat circle factor, and the H-flux component along $\mathcal{H}^1(N)$ survives T-duality along any such factor. □

Remark 3.3. (Relationship with topological T-duality). In the framework of Bouwknegt–Evslin–Mathai [5], T-duality along a circle bundle $\pi\colon E \to X$ exchanges the first Chern class $c_1(E)$ with the fiberwise integral $\int_{S^1} \hat{H}$ of the dual H-flux. In the product setting $M = \Sigma_g \times N \times S^1$, the relevant fiberwise integral is

$$\int_{S^1} H \;=\; \mathrm{vol}_{\Sigma_g} \otimes \int_{S^1} \beta.$$

Writing $\beta = \gamma + c\,d\theta$ with $\gamma \in \mathcal{H}^1(N)$ and $c \in \mathbb{R}$, one has $\int_{S^1} \beta = c \cdot \mathrm{length}(S^1)$. Three cases arise (assuming $\mathcal{P}_1(N) = \{0\}$):

(a) If $\gamma = 0$ and $c \neq 0$ (i.e. $r^\sharp = 0$): the BEM formula gives $c_1(\hat{E}) \neq 0$ and the entire H-flux is absorbed into the topology of the dual bundle.
(b) If $\gamma \neq 0$ and $c = 0$ (i.e. $r^\sharp = 1$): the BEM formula gives $c_1(\hat{E}) = 0$, no topological change occurs, and the full H-flux persists in the dual background.
(c) If $\gamma \neq 0$ and $c \neq 0$ (i.e. $r^\sharp = 1$): the BEM formula gives $c_1(\hat{E}) \neq 0$ (topological change), but the H-flux component $\mathrm{vol}_{\Sigma_g} \wedge\gamma$ survives as H-flux in the dual background (Proposition 3.1(ii)). In this case, T-duality simultaneously produces a topological change *and* leaves residual H-flux.

Thus $r^\sharp$ does not determine the fiberwise integral $\int_{S^1} H$ directly; rather, it detects whether the H-flux has an *irreducible component* $\gamma \in \mathcal{H}^1(N) \setminus \{0\}$ that survives T-duality regardless of whether the fiberwise integral vanishes or not. The additional content of Theorem 3.2 beyond [5] is twofold: the identification of the surviving component via the parallel-form strata, and the preservation of holonomic irreducibility on the reduced product, neither of which is visible in the purely topological BEM framework.

We note that the BEM framework operates with integral cohomology classes $[H] \in H^3(M;\mathbb{Z})$, whereas $r^\sharp$ is defined via de Rham cohomology. The criterion $\gamma \neq 0$ versus $\gamma = 0$ is independent of the coefficient ring.

3.1. **Multiple T-dualities and the irreducible kernel.** We now extend the analysis to successive T-dualities along multiple circle factors. Let $M = \Sigma_g \times N \times T^k$ with product metric $g = g_\Sigma \oplus g_N \oplus g_{T^k}$, where (N, g_N) is a compact oriented Riemannian manifold with $b_1(N) \geq 1$, and $T^k = S^1_{\theta_1} \times \cdots \times S^1_{\theta_k}$ is a flat k-torus with coordinates $(\theta_1, \ldots, \theta_k)$. The H-flux is $H =$

$\mathrm{vol}_{\Sigma_g} \wedge \beta$ of pure bidegree $(2,1)$ with $\beta \in \mathcal{H}^1(N \times T^k)$. We decompose

$$\beta = \gamma + \sum_{i=1}^{k} c_i \, d\theta_i, \qquad \gamma \in \mathcal{H}^1(N), \quad c_i \in \mathbb{R}. \tag{3.3}$$

The key observation is that, for H-flux of pure bidegree $(2,1)$, the obstruction to successive T-dualities identified by Bouwknegt–Evslin–Mathai [5] vanishes automatically, and the dualities are non-interfering.

Lemma 3.4 (Vanishing of the BEM obstruction)**.** *Under the hypotheses above,* $\int_{T^2_{ij}} H = 0$ *for every* 2*-subtorus* $T^2_{ij} = S^1_{\theta_i} \times S^1_{\theta_j} \subset T^k$ *with* $i \neq j$*.*

Proof. Since $H = \mathrm{vol}_{\Sigma_g} \wedge \beta$ has exactly two indices along Σ_g and one index along $N \times T^k$, the only non-vanishing components of H are $H_{\alpha\beta a}$ with α, β tangent to Σ_g and a tangent to $N \times T^k$. In particular, H has no component with two or more indices along T^k. Therefore, the restriction of H to any 2-subtorus $T^2_{ij} \subset T^k$ vanishes identically, and $\int_{T^2_{ij}} H = 0$. □

Lemma 3.5 (Non-interference of successive T-dualities)**.** *After Buscher T-duality along* $S^1_{\theta_i}$*, the dual background satisfies* $\widetilde{G}_{\theta_j\theta_j} = 1$ *and* $\widetilde{G}_{\mu\theta_j} = 0$ *for all* $j \neq i$ *and all* $\mu \neq \theta_j$*. In particular, the Buscher rules for T-duality along* $S^1_{\theta_j}$ *apply to the dual background in the same form as to the original product metric.*

Proof. We show that $B_{\theta_i\theta_j} = 0$ for all $i \neq j$. Since $H = dB$ and H has no component with two or more indices along T^k (as established in the proof of Lemma 3.4), we have

$$H_{\theta_i\theta_j\mu} = \partial_{\theta_i} B_{\theta_j\mu} - \partial_{\theta_j} B_{\theta_i\mu} + \partial_\mu B_{\theta_i\theta_j} = 0$$

for all μ. On the product metric, B is independent of the torus coordinates: $\partial_{\theta_i} B_{\theta_j\mu} = \partial_{\theta_j} B_{\theta_i\mu} = 0$. Thus $\partial_\mu B_{\theta_i\theta_j} = 0$ for all μ, so $B_{\theta_i\theta_j}$ is constant on M. Since the B-field admits gauge transformations $B \to B + \omega$ for any closed 2-form ω, choosing $\omega = -B_{\theta_i\theta_j} \, d\theta_i \wedge d\theta_j$ (which is closed on T^k) sets $B_{\theta_i\theta_j} = 0$.

Applying the Buscher rules [3, 4] for T-duality along $S^1_{\theta_i}$ on a product metric with $G_{\theta_i\theta_i} = 1$ and $G_{\mu\theta_i} = 0$, the dual metric components involving θ_j $(j \neq i)$ are:

$$\widetilde{G}_{\theta_j\theta_j} = G_{\theta_j\theta_j} - \frac{G^2_{\theta_j\theta_i} - B^2_{\theta_j\theta_i}}{G_{\theta_i\theta_i}} = 1 - \frac{0-0}{1} = 1,$$

$$\widetilde{G}_{\mu\theta_j} = G_{\mu\theta_j} - \frac{G_{\mu\theta_i} \, G_{\theta_i\theta_j} - B_{\mu\theta_i} \, B_{\theta_i\theta_j}}{G_{\theta_i\theta_i}} = 0 \quad \text{for all } \mu \neq \theta_j.$$

For the dual B-field: $\widetilde{B}_{\theta_j\theta_i} = G_{\theta_j\theta_i}/G_{\theta_i\theta_i} = 0$, and $\widetilde{B}_{\mu\theta_j} = B_{\mu\theta_j} - (G_{\mu\theta_i} \, B_{\theta_j\theta_i} - B_{\mu\theta_i} \, G_{\theta_j\theta_i})/G_{\theta_i\theta_i} = B_{\mu\theta_j}$ for $\mu \neq \theta_i$. Thus the dual background has the same product structure in the θ_j-direction as the original, and the Buscher rules for $S^1_{\theta_j}$ apply without modification. □

We can now state the main result of this subsection.

Theorem 3.6 (Irreducible kernel of the H-flux). *Let $M = \Sigma_g \times N \times T^k$ with product metric, $\mathcal{P}^1(N) = \{0\}$, and $H = \mathrm{vol}_{\Sigma_g} \wedge \beta$ of pure bidegree $(2,1)$ with $\beta \in \mathcal{H}^1(N \times T^k)$. Write $\beta = \gamma + \sum_{i=1}^k c_i\, d\theta_i$ with $\gamma \in \mathcal{H}^1(N)$ and $c_i \in \mathbb{R}$. Then:*

(i) *For every subset $I \subseteq \{1, \ldots, k\}$, the composed T-duality $T_I := \prod_{i \in I} T_{\theta_i}$ along the circle factors $\{S^1_{\theta_i}\}_{i \in I}$ is well-defined and independent of the order of composition.*

(ii) *The H-flux in the T_I-dual background is*
$$\widetilde{H}_I = \mathrm{vol}_{\Sigma_g} \wedge \Big(\gamma + \sum_{i \notin I} c_i\, d\theta_i\Big).$$

(iii) *The H-flux after T-duality along the full torus T^k (i.e. $I = \{1, \ldots, k\}$) is*
$$\widetilde{H}_{\{1,\ldots,k\}} = \mathrm{vol}_{\Sigma_g} \wedge \gamma.$$

(iv) *$r^\sharp = 1$ if and only if $\gamma \neq 0$, if and only if the H-flux cannot be completely eliminated by any composed T-duality T_I along circle factors of T^k.*

Proof. (i) By Lemma 3.4, the BEM obstruction $\int_{T^2_{ij}} H = 0$ vanishes for all $i \neq j$, so successive T-dualities are well-defined (see [5]). By Lemma 3.5, each T-duality preserves the product structure in all other torus directions. Since the Buscher rules for $S^1_{\theta_i}$ act only on the components of B with a θ_i-index, and by Lemma 3.5 the components with a θ_j-index ($j \neq i$) are unaffected, the result is independent of the order.

(ii) We proceed by induction on $|I|$. The base case $|I| = 1$ is Proposition 3.1. For the inductive step, suppose the result holds for I with $|I| = m < k$, and let $j \notin I$. By the inductive hypothesis, the H-flux after T_I is $\widetilde{H}_I = \mathrm{vol}_{\Sigma_g} \wedge (\gamma + \sum_{i \notin I} c_i\, d\theta_i)$. By Lemma 3.5, the T_I-dual background has product structure in the θ_j-direction ($\widetilde{G}_{\theta_j\theta_j} = 1$, $\widetilde{G}_{\mu\theta_j} = 0$), so Proposition 3.1 applies: T-duality along $S^1_{\theta_j}$ converts the component $c_j\, d\theta_j$ into geometric flux and leaves the remaining components unchanged. Thus $\widetilde{H}_{I \cup \{j\}} = \mathrm{vol}_{\Sigma_g} \wedge (\gamma + \sum_{i \notin I \cup \{j\}} c_i\, d\theta_i)$.

(iii) This is the case $I = \{1, \ldots, k\}$ of (ii).

(iv) Since $\mathcal{P}^1(N) = \{0\}$, the parallel-form stratum satisfies $\mathcal{P}^1(N \times T^k) = \mathcal{P}^1(N) \oplus \bigoplus_{i=1}^k \mathbb{R} \cdot d\theta_i = \bigoplus_{i=1}^k \mathbb{R} \cdot d\theta_i$. The condition $r^\sharp = 0$ with $r_{2,1} = 1$ means $\beta \in \mathcal{P}^1(N \times T^k)$, i.e. $\gamma = 0$ and $\beta = \sum_i c_i\, d\theta_i$. By (iii), $\widetilde{H}_{\{1,\ldots,k\}} = 0$: the H-flux is completely eliminated. Conversely, $r^\sharp = 1$ means $\beta \notin \mathcal{P}^1(N \times T^k)$, which forces $\gamma \neq 0$. By (iii), $\widetilde{H}_{\{1,\ldots,k\}} = \mathrm{vol}_{\Sigma_g} \wedge \gamma \neq 0$. Since this is the H-flux after T-duality along *all* circle factors, and (ii) shows that partial

T-dualities leave at least the γ-component, no composed T-duality T_I can eliminate the H-flux entirely. □

Remark 3.7 (Connection to the flux chain)**.** In the framework of Shelton–Taylor–Wecht [8], successive T-dualities convert H-flux into geometric flux (f-flux), then into non-geometric Q-flux and R-flux. The component $\sum_{i\in I} c_i\, d\theta_i$ of β enters this chain through the T-dualities T_I: each $c_i\, d\theta_i$ is converted first into geometric flux (Proposition 3.1) and may be further converted by subsequent dualities. The component $\gamma \in \mathcal{H}^1(N)$, by contrast, cannot enter the chain at all: it has no component along any torus direction, so no T-duality along a flat circle factor can act on it. This component is the *irreducible kernel* of the H-flux—the part that remains H-flux in every duality frame. When $r^\sharp = 0$, the irreducible kernel vanishes and the entire H-flux is accessible to the chain. When $r^\sharp = 1$, the irreducible kernel is non-trivial: the H-flux can never be fully converted into geometric or non-geometric flux by T-dualities along flat circle factors.

4. Examples

Example 4.1. Let $M = \Sigma_g \times \Sigma_{g'} \times T^2$ with $g, g' \geq 2$, product metric, and $T^2 = S^1_\phi \times S^1_\theta$. Since $\mathrm{Ric}(\Sigma_{g'}) = K_{g'}\, g_{\Sigma_{g'}}$ with $K_{g'} < 0$, we have $\mathcal{P}^1(\Sigma_{g'}) = \{0\}$. Let $\beta = \gamma + c_\phi\, d\phi + c_\theta\, d\theta$ with $\gamma \in \mathcal{H}^1(\Sigma_{g'})$.

- (i) If $\gamma \neq 0$: $r^\sharp = 1$. T-duality along S^1_ϕ converts $c_\phi\, d\phi$ into geometric flux; T-duality along S^1_θ converts $c_\theta\, d\theta$. After both T-dualities, $\widetilde{H} = \mathrm{vol}_{\Sigma_g} \wedge \gamma \neq 0$. The irreducible kernel persists.
- (ii) If $\gamma = 0$: $r^\sharp = 0$. After T-duality along both circles, $\widetilde{H} = 0$. The H-flux is completely converted.

Example 4.2 (Hyperbolic three-manifold)**.** Let N^3 be a compact hyperbolic three-manifold with $b_1(N^3) \geq 1$ (for instance, a mapping torus of a pseudo-Anosov diffeomorphism of a surface, which is hyperbolic by Thurston's theorem). Since $\mathrm{Ric}(N^3) = -2\, g_{N^3}$, a parallel vector field X on N^3 would satisfy $\mathrm{Ric}(X, X) = -2|X|^2 = 0$, forcing $X = 0$. Thus $\mathcal{P}_1(N^3) = \{0\}$.

On $M = \Sigma_g \times N^3 \times S^1$ with any non-zero $\beta \in \mathcal{H}^1(N^3)$: $r^\sharp = 1$, the H-flux survives T-duality, and the connection is irreducible on both M and on the reduced product $\Sigma_g \times N^3$.

Example 4.3 (Flat torus factor)**.** Let $N = T^2$ with the flat metric, so that $\mathcal{P}_1(T^2) = \mathcal{H}^1(T^2) = \mathbb{R}^2$. Then $\mathcal{P}_1(T^2 \times S^1) = \mathbb{R}^3 = \mathcal{H}^1(T^2 \times S^1)$: every harmonic 1-form is parallel. For any non-zero $\beta \in \mathcal{H}^1(T^2 \times S^1)$: $r^\sharp = 0$. There exists a specific circle factor S^1_β (identified by the parallel vector field $\beta^\sharp$) such that T-duality along S^1_β converts the H-flux entirely into geometric flux. The connection on $\Sigma_g \times T^2 \times S^1$ is irreducible (by [1, Corollary 5.3], since $r = 1$), but this irreducibility is not resistant to T-duality along S^1_β.

Example 4.4 (Mixed parallel stratum)**.** Let $N = \Sigma_{g'} \times S^1_\varphi$ with $g' \geq 2$ and the product metric. Then $\mathcal{P}_1(N) = \{0\} \oplus \mathbb{R} \cdot d\varphi = \mathbb{R} \cdot d\varphi$ and $\mathcal{P}_1(N \times S^1) = \mathbb{R} \cdot d\varphi \oplus \mathbb{R} \cdot d\theta$. On $M = \Sigma_g \times \Sigma_{g'} \times S^1_\varphi \times S^1$:

If $\beta \in \mathcal{H}^1(\Sigma_{g'})$: $\beta \notin \mathcal{P}_1(N \times S^1)$, $r^\sharp = 1$. The H-flux survives T-duality along both S^1_φ and S^1.

If $\beta = d\varphi$: $r^\sharp = 0$. T-duality along S^1_φ converts the H-flux into geometric flux. But T-duality along S^1 (the other circle) does not, because $d\varphi$ has no component along $d\theta$.

If $\beta = d\theta$: $r^\sharp = 0$. T-duality along S^1 converts the H-flux. But T-duality along S^1_φ does not.

This example illustrates that $r^\sharp$ discriminates among cohomology classes on the same manifold: the three choices of β produce qualitatively different T-duality behaviour, yet all have $r = 1$.

5. Discussion

The topological T-duality programme of Bouwknegt, Evslin, and Mathai [5], and its subsequent developments by Bunke–Schick [6], and Waldorf [9], determines the topology and H-flux of the T-dual from the class $[H] \in H^3(M; \mathbb{Z})$ and the first Chern class of the circle bundle, without reference to the Riemannian metric. As shown in Remark 3.3, the invariant $r^\sharp$ determines the vanishing of the fiberwise integral $\int_{S^1} H$ that enters the BEM formula, thus predicting the topological outcome from the position of the mixed class relative to the parallel-form strata. This constitutes a metric refinement: $r^\sharp$ uses the Riemannian structure (via $\mathcal{P}_1$) to extract information that is invisible to the purely topological framework of [5], namely whether the cohomological coupling is aligned with the flat sub-factors of the de Rham splitting.

When the H-flux is transverse to the parallel-form strata ($r^\sharp = 1$), it is resistant to T-duality along any flat circle factor and the holonomy irreducibility of the associated connection is preserved under dimensional reduction. When the H-flux is absorbed by the parallel-form strata ($r^\sharp = 0$), there exists a specific circle factor along which T-duality converts the entire H-flux into geometric flux, and the holonomy irreducibility does not survive dimensional reduction along that factor.

This distinction is invisible to topological T-duality, which operates at the level of cohomology classes and does not see the parallel-form strata. It is equally invisible to the topological invariant $r = \operatorname{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}})$, which equals 1 in all cases considered. The metric information encoded in $\dim \mathcal{K}$, and hence in $r^\sharp$, is essential for predicting the T-duality behaviour.

When M_2 admits multiple circle factors, Theorem 3.6 sharpens the single-duality analysis: for H-flux of pure bidegree $(2, 1)$ on $M = \Sigma_g \times N \times T^k$, the Bouwknegt–Evslin–Mathai obstruction to successive T-dualities vanishes automatically (Lemma 3.4), the dualities are non-interfering (Lemma 3.5), and their combined effect decomposes the H-flux into a convertible part $\sum c_i\, d\theta_i$ (accessible to the Shelton–Taylor–Wecht flux chain [8]) and an irreducible kernel $\mathrm{vol}_{\Sigma_g} \wedge \gamma$ that persists as H-flux in every duality frame. The invariant $r^\sharp$ detects whether this kernel is non-trivial: $r^\sharp = 1$ if and only if $\gamma \neq 0$. The test reduces to verifying membership of β in $\mathcal{P}_1(N \times T^k)$, a linear-algebraic computation on finite-dimensional spaces. The vanishing of the BEM obstruction is specific to the bidegree $(2,1)$ condition and does not hold for generic H-flux, giving the bidegree hypothesis a structural role in the theory of multiple T-dualities.

More broadly, the results of this note reveal a structural connection between two a priori independent phenomena: the irreducibility of holonomy for metric connections with torsion (governed by $r^\sharp$ via [1]) and the survival of H-flux under T-duality (governed by the Buscher rules). The invariant $r^\sharp$ provides a unified explanation: in both cases, the relevant dichotomy is whether the cohomological coupling between the factors is transverse to the de Rham flat sub-factors ($r^\sharp = 1$) or aligned with them ($r^\sharp = 0$).

We note that this dichotomy has an exact structural parallel in the Kaluza–Klein analysis of spin–orbit-coupled Bose–Einstein condensates carried out in [2]. In that setting, $r^\sharp = 0$ at the product metric means that the harmonic 1-form $\beta = c^{(+)} d\varphi_+ + c^{(-)} d\varphi_-$ belongs to $\mathcal{P}_1$, identifying a circle direction S^1_β in the phase space along which a phase-locking protocol eliminates the topological obstruction; $r^\sharp = 1$ at the physical Kaluza–Klein metric certifies that no such protocol exists. The mechanism is the same in both contexts: $r^\sharp = 0$ identifies a flat circle factor, read off from the parallel vector field $\beta^\sharp$, along which the cohomological coupling can be neutralised (by T-duality or dimensional reduction in the present setting, by phase-locking in [2]), while $r^\sharp = 1$ certifies that no such factor exists.

A natural direction for future investigation is to extend this analysis to non-product metrics (Kaluza–Klein backgrounds), where the Buscher rules involve additional terms from the off-diagonal metric components. In the context of flux compactifications, the distinction between H-flux and geometric flux—which $r^\sharp$ detects on product geometries—determines the structure of the effective superpotential and the pattern of moduli stabilisation [8]. The irreducible kernel identified by Theorem 3.6 represents the component of the H-flux whose contribution to the effective potential is invariant across all duality frames accessible by T-duality along flat circle factors; extending $r^\sharp$ to fibered geometries could provide new tools for classifying string vacua and for identifying duality-invariant sectors of the flux landscape.

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