Local Topological Constraints on Berry Curvature in Spin–Orbit Coupled BECs

Alexander Pigazzini, Magdalena Toda

Abstract

We establish a local topological obstruction to the simultaneous flattening of Berry curvature in spin–orbit-coupled Bose–Einstein condensates (SOC BECs), which remains valid even when the global Chern number vanishes. For a generic two-component SOC BEC, the extended parameter space is the total space M of a principal $U(1)_+ \times U(1)_-$ bundle over the Brillouin torus T^2_{BZ}. On M, we construct a Kaluza–Klein metric and a natural metric connection ∇^C whose torsion 3-form encodes the synthetic gauge fields.

Under the physically relevant assumption of constant Berry curvatures, the harmonic part of this torsion defines a mixed cohomology class

$$[\omega] \in \left(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+})\right) \oplus \left(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_-})\right)$$

with mixed tensor rank $r = 1$. By adapting the *Pigazzini–Toda (PT) lower bound* to the Kaluza–Klein setting through explicit pointwise curvature analysis, we demonstrate that the obstruction kernel $\mathcal{K}$ vanishes for the physical metric, yielding the sharp inequality $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \geq 1$. This bound forces the existence of at least one off-diagonal curvature operator, preventing the complete gauging-away of Berry phases even in regimes with zero net topological charge. This work provides the first cohomological lower bound, based on the PT framework, certifying locally irremovable curvature in SOC BECs beyond the Chern-number paradigm.

Keywords: Berry curvature; spin–orbit-coupled Bose–Einstein condensates; synthetic gauge fields; holonomy with torsion; mixed cohomology; Kaluza–Klein geometry.
MSC (2020): 53C29, 53C80, 81V45, 82B10, 51P05, 55S20, 57R19, 57R56, 58A12.

1 Introduction

The study of spin–orbit-coupled (SOC) Bose–Einstein condensates (BECs) stands at a rich confluence of differential geometry, quantum many-body physics, and synthetic gauge theory. This interdisciplinary effort is part of the broader pursuit of artificial gauge fields in engineered quantum systems, spanning ultracold atoms, photonics, and solid-state platforms [1]. Advances in generating artificial gauge fields with Raman lasers have made these systems a versatile laboratory for exploring Berry curvature, synthetic magnetic fields, and non-Abelian textures with unprecedented control [6, 7, 14]. Experimental realizations in laboratories such as NIST [9] have demonstrated the ability to engineer synthetic spin-orbit coupling in ultracold atomic gases, providing a controllable platform for investigating topological phenomena in quantum matter.

A persistent theoretical challenge is to identify *local* topological obstructions that constrain the Berry curvature even when global invariants—such as the first Chern number—vanish. While the Chern number $c_1 = \frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F$ quantizes the net flux through the Brillouin zone, its vanishing does not guarantee that the curvature F can be locally flattened everywhere; geometric obstructions may prevent the simultaneous annihilation of curvature components along independent directions in the parameter space. Understanding such local constraints is crucial for assessing the robustness of topological features in quantum systems and for designing protocols that exploit geometry beyond global topological charges.

This work addresses that challenge by bringing a recent geometric result into physical focus. In [12], a lower bound was established for the off-diagonal holonomy of metric connections with totally skew-symmetric torsion on product manifolds. The bound is expressed in terms of a *mixed* deRham cohomology class $[\omega] \in H^p(M_1) \otimes H^q(M_2)$ associated with the harmonic part of the torsion 3-form:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \;\geq\; r^\sharp := \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}, \tag{1.1}$$

where $r = \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}})$ is the minimal number of simple tensors needed to represent the mixed cohomology class, $\mathcal{K}$ is an obstruction kernel encoding the intersection of this class with spaces of parallel forms, and $\mathfrak{hol}^{\mathrm{off}}(\nabla^C)$ denotes the off-diagonal part of the holonomy algebra that mixes the two factor manifolds. This bound is a topological invariant that persists under metric deformations preserving the parallel-form strata.

Methodological adaptation: From product to Kaluza–Klein geometry

While Theorem5.2 of [12] is formally stated for Riemannian product metrics, the physical metric g_M in our SOC BEC model is a Kaluza–Klein metric induced by the synthetic gauge fields $A^{(\pm)}$. *We do not apply the theorem as a black box.* Instead, we extend its reach to the present geometry through a three-step strategy:

(i) Deformation. We construct a smooth one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ that interpolates between a Riemannian product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ at $\varepsilon = 0$ and the physical Kaluza–Klein metric $g_M = g_1$ at $\varepsilon = 1$ (Section4). This deformation is achieved by continuously introducing the gauge potentials $A^{(\pm)}$ into the fiber metric via the connection 1-forms $\Theta_\varepsilon^{(\pm)} = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$.

(ii) Direct verification for $\varepsilon > 0$. For every $\varepsilon > 0$, we prove that the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes (Theorem3.14), yielding a non-trivial reduced rank $r^\sharp_\varepsilon = 1$. By explicit computation in an adapted gauge (AppendixB), we evaluate the curvature $R^{C_\varepsilon}(X, Z)$ for mixed inputs $X \in \mathcal{H}_p$, $Z \in \mathcal{V}_p$ and show that its off-diagonal projection $\pi_{\mathrm{off}}(R^{C_\varepsilon}(X, Z)) \neq 0$ at each point $p \in M$, demonstrating that at least one linearly independent off-diagonal curvature operator exists in the holonomy algebra. This establishes the bound $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ for all $\varepsilon \in (0, 1]$ through a direct analysis of the curvature tensor, without relying on continuity arguments.

(iii) Application to the physical metric. Since the bound holds for every $\varepsilon \in (0, 1]$ and the physical connection corresponds to $\varepsilon = 1$, we obtain the desired inequality for the physical system. The triviality of the bound at $\varepsilon = 0$ (where $r^\sharp_0 = 0$ due to complete

absorption by parallel forms in the product geometry) highlights the essential role of the Kaluza–Klein coupling in generating irreducible topological constraints.

We take as the extended parameter space the smooth manifold

$$M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-},$$

where T^2_{BZ} is the Brillouin torus parametrized by crystal momenta (k_x, k_y), and $S^1_{\phi_+}$, $S^1_{\phi_-}$ represent the global U(1) phase (associated with particle-number conservation) and the relative U(1) phase between the two spin components, respectively. Our analysis shows that under the physically motivated assumption of constant Berry curvatures $F^{(\pm)} = c^{(\pm)}\mathrm{vol}_{\mathrm{BZ}}$ (Assumption2.3), the mixed tensor rank is $r = 1$. The vanishing of the obstruction kernel for the Kaluza–Klein geometry (Theorem2.14) then yields the sharp inequality:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \geq 1 \quad \text{for every point } p \in M. \tag{1.2}$$

This lower bound implies that momentum degrees of freedom are fundamentally "locked" to the phase sectors in a topologically robust manner: at least one independent off-diagonal curvature operator persists, mixing the Brillouin zone with the phase directions even when the total Chern number $c_1^{(+)} + c_1^{(-)}$ vanishes. Such mixing cannot be removed by any smooth gauge transformation or metric deformation that preserves the cohomology classes $[F^{(\pm)}]$. This result reveals a *local* topological obstruction invisible to the Chern number and suggests new interferometric protocols to detect these geometric constraints through measurements of correlated Berry phases in the two U(1) sectors.

Organization of the paper

The paper is organized as follows. Section2 constructs the Kaluza–Klein metric and defines the metric connection with torsion encoding the synthetic gauge fields. Section3 analyzes the mixed cohomology structure, computes the tensor rank, and determines the obstruction kernel for both the product and deformed metrics. Section4 presents the deformation family and establishes the non-trivial lower bound through direct curvature analysis. Section5 interprets the bound in the context of SOC BECs and discusses its physical consequences. Section7 provides illustrative examples, including the paradigmatic case of vanishing total Chern flux. Finally, Section9 summarizes the results and outlines directions for future work. *Technical details on the Levi-Civita connection and the explicit curvature computation are relegated to AppendicesA andB, respectively.*

2 Geometric Construction of the Extended Parameter Space

A generic two-component spin–orbit-coupled Bose–Einstein condensate (SOC BEC) confined in a quasi-two-dimensional toroidal trap is characterized by three independent, experimentally accessible degrees of freedom: the crystal momenta (k_x, k_y) spanning a Brillouin zone, a global $U(1)$ phase ϕ_+ associated with particle-number conservation, and a relative $U(1)$ phase ϕ_- between the two internal spin states [9, 11].

The natural mathematical framework for this system is a *principal bundle* with structure group $U(1)_+ \times U(1)_-$, whose total space M serves as the extended parameter space. Topologically, M is diffeomorphic to the four-dimensional torus T^4, but its geometric structure is that of a Kaluza–Klein manifold when equipped with the synthetic gauge fields.

2.1 Global geometric structure and the product approximation

For the purpose of applying the PT lower bound, we work in a framework where the parameter space admits a global product structure, possibly up to a smooth deformation. Although the parameter space $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ is a global topological product, the Berry curvatures $F^{(\pm)}$ are associated to the quantum eigenbundles over T^2_{BZ}, rather than to line bundles over M. Consequently, non-vanishing Chern classes $c_1^{(\pm)} \neq 0$ are fully compatible with the product structure of M.

Definition 2.1 (Extended parameter space as a product manifold)**.** *We consider the smooth manifold*

$$M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-} \cong T^4,$$

equipped with a family of Riemannian metrics $\{g_\varepsilon\}_{\varepsilon \in [0,1]}$ *such that:*

1. *For* $\varepsilon = 0$, $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ *is a Riemannian product metric.*
2. *For* $\varepsilon = 1$, $g_1 = g_M$ *is the physical Kaluza–Klein metric given by* (2.1)*.*
3. *The family is smooth in* ε *and preserves the orthogonal splitting* $TM = \mathcal{H}_\varepsilon \oplus \mathcal{V}$*, where* $\mathcal{V} = \ker d\pi$ *is the vertical distribution (independent of* ε*) and* $\mathcal{H}_\varepsilon$ *is the* g_ε*-orthogonal complement.*

Remark 2.2 (Topological vs. geometric product structure)**.** *While the underlying smooth manifold is a product, the physical metric* g_M *is not a Riemannian product unless the connections* $A^{(\pm)}$ *vanish. The family* $\{g_\varepsilon\}$ *interpolates between the geometric product structure (*$\varepsilon = 0$*) and the physical Kaluza–Klein structure (*$\varepsilon = 1$*). This deformation is essential to apply the PT theorem, which requires a genuine Riemannian product structure.*

2.2 Constant Berry curvature assumption

To make the geometric analysis fully explicit and to cover a wide range of experimentally relevant configurations, we work under the following assumption:

Assumption 2.3 (Constant Berry curvatures)**.** *The Berry curvature 2-forms are constant multiples of the Brillouin-zone volume form:*

$$F^{(\pm)} = dA^{(\pm)} = c^{(\pm)}\,\mathrm{vol}_{\mathrm{BZ}}, \qquad c^{(\pm)} \in \mathbb{R} \setminus \{0\},$$

where $\mathrm{vol}_{\mathrm{BZ}} = dk_x \wedge dk_y$*. The constants* $c^{(\pm)}$ *are related to the Chern numbers by*

$$c_1^{(\pm)} = \frac{1}{2\pi} \int_{T^2_{\mathrm{BZ}}} F^{(\pm)} = \frac{\mathrm{Area}(T^2_{\mathrm{BZ}})}{2\pi}\, c^{(\pm)}.$$

This condition is satisfied in numerous SOC BEC realizations, including linear Rashba–Dresselhaus couplings and uniform synthetic magnetic fields.

Remark 2.4 (Topological invariance)**.** *The cohomology classes* $[F^{(\pm)}] \in H^2(T^2_{\mathrm{BZ}}; \mathbb{R})$ *are topological invariants. Under Assumption2.3, these classes are proportional:* $[F^{(-)}] = \lambda [F^{(+)}]$ *with* $\lambda = c^{(-)}/c^{(+)}$*. Any deformation of the connection within its cohomology class preserves these invariants, ensuring the robustness of our results.*

2.3 The Kaluza–Klein metric and its deformation to a product

We construct a family of Riemannian metrics on M that interpolates between the physical Kaluza–Klein metric and a product metric. Let $g_{\mathrm{BZ}} = dk_x^2 + dk_y^2$ be the flat metric on the base torus T^2_{BZ}.

Definition 2.5 (Deformation family of Kaluza–Klein metrics)**.** *For $\varepsilon \in [0,1]$, define the metric on M by:*

$$g_\varepsilon = \pi^* g_{\mathrm{BZ}} + \big(d\phi_+ + \varepsilon\, \pi^* A^{(+)}\big)^2 + \big(d\phi_- + \varepsilon\, \pi^* A^{(-)}\big)^2, \tag{2.1}$$

where $\pi : M \to T^2_{\mathrm{BZ}}$ is the projection onto the first factor. For $\varepsilon = 0$ we obtain the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, and for $\varepsilon = 1$ we obtain the physical Kaluza–Klein metric $g_1 = g_M$.

Remark 2.6 (Global well-definedness of the metric family)**.** *The metrics g_ε are* globally well-defined *on the total space M of the principal $U(1)_+ \times U(1)_-$-bundle over T^2_{BZ}. The 1-forms $A^{(\pm)}$ are connection 1-forms on local trivializations; the quantities $(d\phi_\pm + \varepsilon\pi^* A^{(\pm)})^2$ are* invariant under gauge transformations *$A^{(\pm)} \to A^{(\pm)} + d\lambda^{(\pm)}$ because the fiber coordinates $\phi_\pm$ transform as $\phi_\pm \to \phi_\pm - \varepsilon\lambda^{(\pm)}$. Consequently, the metric g_ε descends to a well-defined Riemannian metric on M even when the curvatures $F^{(\pm)}$ have non-zero Chern classes.*

Theorem 2.7 (Properties of the deformation family)**.** *The family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ defined on M satisfies the following properties:*

(i) Regularity: Each g_ε is a smooth, positive-definite Riemannian metric on M. The projection $\pi : (M, g_\varepsilon) \to (T^2_{\mathrm{BZ}}, g_{\mathrm{BZ}})$ is a Riemannian submersion for every $\varepsilon \in [0,1]$.

(ii) Orthogonality: The vertical distribution $\mathcal{V} = \ker d\pi = \mathrm{span}\{\partial_{\phi_+}, \partial_{\phi_-}\}$ is g_ε-orthogonal to the horizontal distribution $\mathcal{H}_\varepsilon$, where the latter is defined as the kernel of the connection 1-forms $\Theta^{(\pm)}_\varepsilon = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$.

(iii) Fiber Isotropy: Each fibre $\pi^{-1}(k) \cong S^1 \times S^1$ inherits a flat metric $g_{\text{fibre}} = d\phi_+^2 + d\phi_-^2$ which is independent of both the base point $k \in T^2_{\mathrm{BZ}}$ and the deformation parameter ε.

(iv) Geodesic Geometry: The fibres are totally geodesic submanifolds of (M, g_ε) for every ε, and the horizontal distribution $\mathcal{H}_\varepsilon$ has a curvature proportional to $\varepsilon F^{(\pm)}$.

Proof. (i)-(ii) By construction, $g_\varepsilon = \pi^* g_{\mathrm{BZ}} + (\Theta^{(+)}_\varepsilon)^2 + (\Theta^{(-)}_\varepsilon)^2$. Since g_{BZ} is positive definite and the vertical forms are linearly independent, g_ε is a valid Riemannian metric. The orthogonality follows from the fact that $g_\varepsilon(X, V) = 0$ whenever $X \in \mathcal{H}_\varepsilon$ (so $\Theta^{(\pm)}_\varepsilon(X) = 0$) and $V \in \mathcal{V}$ (so $d\pi(V) = 0$).

(iii) Restricting g_ε to the vertical distribution $\mathcal{V}$ means evaluating it on vectors V such that $d\pi(V) = 0$. In this case, $\pi^* A^{(\pm)}(V) = A^{(\pm)}(d\pi(V)) = 0$, so $\Theta^{(\pm)}_\varepsilon|_{\mathcal{V}} = d\phi_\pm$. Thus $g_\varepsilon|_{\mathcal{V}} = d\phi_+^2 + d\phi_-^2$, which is constant and independent of ε.

(iv) The fibers are flat tori, and the connection potentials $A^{(\pm)}$ depend only on the base coordinates. The vanishing of the second fundamental form of the fibers follows from the Kaluza-Klein structure with constant fiber metrics. The curvature of the horizontal distribution is given by $d\Theta^{(\pm)}_\varepsilon = \varepsilon dA^{(\pm)} = \varepsilon F^{(\pm)}$, confirming that ε scales the synthetic magnetic field without altering the fiber geometry. □

2.4 Metric connection with totally skew-symmetric torsion

For each $\varepsilon \in [0,1]$, we define a metric connection ∇^{C_ε} with torsion that encodes the physical Berry curvature.

Definition 2.8 (Torsion 3-form family). *For each ε, define the* torsion 3-form*:*

$$T_\varepsilon := F^{(+)} \wedge (d\phi_+ + \varepsilon\pi^* A^{(+)}) + F^{(-)} \wedge (d\phi_- + \varepsilon\pi^* A^{(-)}) \in \Omega^3(M).$$

Under Assumption2.3, this becomes:

$$T_\varepsilon = c^{(+)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_+ + \varepsilon\pi^* A^{(+)}) + c^{(-)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_- + \varepsilon\pi^* A^{(-)}).$$

Remark 2.9 (Pure bigrade structure of the torsion). *With respect to the product decomposition $TM = V_1 \oplus V_2$, where $V_1 = T\,T^2_{\mathrm{BZ}}$ and $V_2 = T(S^1_{\varphi_+} \times S^1_{\varphi_-})$, the torsion T_ε has* pure bigrade$(2,1)$*: every term belongs to $\Gamma(\Lambda^2 V_1^* \otimes V_2^*)$. Indeed, the constituents $F^{(\pm)} \in \Gamma(\Lambda^2 V_1^*)$ and $d\varphi_\pm \in \Gamma(V_2^*)$, while the additional terms $F^{(\pm)} \wedge \varepsilon\pi^* A^{(\pm)}$ would belong to $\Gamma(\Lambda^3 V_1^*)$, which vanishes identically since* $\dim V_1 = 2$.

Consequently, $T_\varepsilon^{1,2} \equiv 0$ as a section for every $\varepsilon \in [0,1]$, verifying the pure bigrade hypothesis of [12, Theorem 5.2]. This ensures that the overlap locus $W^{2,1} \cap W^{1,2}$ is empty and that Case 2 of the PT proof ([12, Theorem 5.2]) is vacuous for the SOCBEC model.

Lemma 2.10 (Cohomological invariance of the torsion class). *For all $\varepsilon \in [0,1]$, the de Rham cohomology class $[T_\varepsilon] \in H^3(M;\mathbb{R})$ is constant and equal to*

$$[T_\varepsilon] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-] \quad \textit{under the Künneth isomorphism.}$$

Proof. We verify that T_ε is closed for every $\varepsilon \in [0,1]$. Using $dF^{(\pm)} = 0$ (curvatures are closed) and $d(d\phi_\pm) = 0$, we compute:

$$dT_\varepsilon = -F^{(+)} \wedge d(\varepsilon\pi^* A^{(+)}) - F^{(-)} \wedge d(\varepsilon\pi^* A^{(-)}) = -\varepsilon \left(F^{(+)} \wedge F^{(+)} + F^{(-)} \wedge F^{(-)} \right).$$

Under Assumption2.3, $F^{(\pm)} = c^{(\pm)} dk_x \wedge dk_y$, hence

$$F^{(\pm)} \wedge F^{(\pm)} = c^{(\pm)2} (dk_x \wedge dk_y) \wedge (dk_x \wedge dk_y) = 0$$

by antisymmetry. Thus $dT_\varepsilon = 0$ for all ε. Under the Künneth decomposition

$$H^3(M;\mathbb{R}) \cong \left(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-}) \right) \oplus \left(H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\phi_+} \times S^1_{\phi_-}) \right),$$

the class $[T_0] = [F^{(+)} \wedge d\phi_+ + F^{(-)} \wedge d\phi_-]$ decomposes as

$$[T_0] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-]$$

in the $(2,1)$-component. The $(1,2)$-component $[T_0]_{1,2} \in H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\varphi_+} \times S^1_{\varphi_-})$ vanishes because every summand ofT_0 has the form $F^{(\pm)} \wedge d\varphi_\pm$, i.e. a 2-form on the base wedged with a 1-form on the fibre; no term of bidegree$(1,2)$ is present. For $T_\varepsilon = T_0 + \varepsilon(F^{(+)} \wedge \pi^* A^{(+)} + F^{(-)} \wedge \pi^* A^{(-)})$, the additional terms are 3-forms of bidegree $(3,0)$ in the Künneth decomposition, being wedge products of forms pulled back from T^2_{BZ}. Since

$$H^3(T^2_{\mathrm{BZ}}) = 0 \quad (\text{as } \dim T^2_{\mathrm{BZ}} = 2 < 3),$$

these terms contribute zero to the cohomology class. Therefore

$$[T_\varepsilon] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-] = [T_0]$$

for all $\varepsilon \in [0,1]$. □

Definition 2.11 (Metric connection family). *For each $\varepsilon \in [0,1]$, define the $(2,1)$-tensor K_ε by:*

$$K_\varepsilon(X,Y,Z) := \frac{1}{2} T_\varepsilon(X,Y,Z), \quad X,Y,Z \in \mathfrak{X}(M).$$

The metric connection with totally skew-symmetric torsion *for parameter ε is:*

$$\nabla^{C_\varepsilon}_X Y := \nabla^{LC_\varepsilon}_X Y + K_\varepsilon(X,Y,\cdot)^{\sharp_\varepsilon},$$

where ∇^{LC_ε} is the Levi-Civita connection of g_ε and $\sharp_\varepsilon$ denotes the musical isomorphism $T^*M \to TM$ induced by g_ε.

Proposition 2.12 (Properties of the connection family). *For each $\varepsilon \in [0,1]$, the connection ∇^{C_ε} satisfies:*

(i) $\nabla^{C_\varepsilon} g_\varepsilon = 0$ (metric compatibility).

(ii) The torsion tensor of ∇^{C_ε} is exactly T_ε, i.e.:

$$\nabla^{C_\varepsilon}_X Y - \nabla^{C_\varepsilon}_Y X - [X,Y] = T_\varepsilon(X,Y,\cdot)^{\sharp_\varepsilon}.$$

(iii) For $\varepsilon > 0$, $\nabla^{LC_\varepsilon} T_\varepsilon \neq 0$ (non-parallel torsion), provided $F^{(\pm)} \neq 0$.

Proof. Properties (i) and (ii) are standard consequences of the definition $\nabla^{C_\varepsilon} = \nabla^{LC_\varepsilon} + \frac{1}{2} T_\varepsilon$: adding a totally skew-symmetric $(2,1)$-tensor to the Levi-Civita connection preserves metric compatibility, and the torsion of the resulting connection equals the skew-symmetric tensor (see, e.g., [15]).

For (iii), AppendixA shows that $\nabla^{LC_\varepsilon} d\phi_\pm = \frac{\varepsilon c^{(\pm)}}{2}(e^2 \otimes e^1 - e^1 \otimes e^2)$, which is nonzero for $\varepsilon > 0$ and $c^{(\pm)} \neq 0$. Since $T_\varepsilon = c^{(+)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_+ + \varepsilon \pi^* A^{(+)}) + c^{(-)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_- + \varepsilon \pi^* A^{(-)})$ and the base form $\operatorname{vol}_{\mathrm{BZ}}$ is parallel, the Leibniz rule gives $\nabla^{LC_\varepsilon} T_\varepsilon \neq 0$. □

2.5 Parallel forms and the obstruction kernel

A crucial ingredient for the PT lower bound is the space of forms that are parallel with respect to the Levi-Civita connection of the product metric ($\varepsilon = 0$) and their behavior under deformation.

Definition 2.13 (Spaces of parallel forms on the factors). *For the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, we define:*

$$\mathcal{P}_2(T^2_{\mathrm{BZ}}) := \{\alpha \in \Omega^2(T^2_{\mathrm{BZ}}) \mid \nabla^{g_{\mathrm{BZ}}} \alpha = 0\} = \mathbb{R} \operatorname{vol}_{\mathrm{BZ}},$$
$$\mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}) := \{\beta \in \Omega^1(S^1_{\phi_+} \times S^1_{\phi_-}) \mid \nabla^{g_{\mathit{fibre}}} \beta = 0\} = \operatorname{span}\{d\phi_+, d\phi_-\}.$$

For $\varepsilon > 0$, we define $\mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$ as the space of 1-forms on M that are g_ε-parallel and vanish on $\mathcal{H}_\varepsilon$ (vertical parallel 1-forms).

Theorem 2.14 (Deformation of vertical parallel 1-forms)**.** *Under Assumption2.3, for the deformation family g_ε, the space of parallel vertical 1-forms $\mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$ satisfies:*

1. *For $\varepsilon > 0$, the space has dimension 1 and consists of constant-coefficient forms $\eta = c_+ d\phi_+ + c_- d\phi_-$ satisfying the algebraic constraint:*

$$c_+ + \lambda c_- = 0, \quad \textit{where } \lambda = \frac{c^{(-)}}{c^{(+)}}.$$

2. *For the product limit $\varepsilon = 0$, the constraint vanishes and the space has dimension 2, being spanned by $\{d\phi_+, d\phi_-\}$.*

Proof. The condition for a 1-form η to be parallel is $\nabla^{LC_\varepsilon}\eta = 0$. For a vertical form $\eta = c_+ d\phi_+ + c_- d\phi_-$, we evaluate the covariant derivative along horizontal directions. As derived in AppendixA using the connection 1-forms $\omega^a_{\ b}$ of the Kaluza–Klein metric g_ε, we have:

$$\nabla^{LC_\varepsilon}\eta = \frac{\varepsilon}{2}(c_+ c^{(+)} + c_- c^{(-)})(e^2 \otimes e^1 - e^1 \otimes e^2).$$

For any $\varepsilon > 0$, the vanishing of this expression requires $c_+ c^{(+)} + c_- c^{(-)} = 0$, which is equivalent to $c_+ + \lambda c_- = 0$. This constraint defines a 1-dimensional subspace. At the point $\varepsilon = 0$, the entire expression vanishes regardless of the coefficients $c_\pm$, reflecting the fact that the Levi-Civita connection of the product metric decouples. This completes the proof (see AppendixA for the component-wise tensor derivation). □

Remark 2.15 (Global extension of vertical parallel forms)**.** *Since the vertical distribution $\mathcal{V} = \ker d\pi$ is* integrable *(being the kernel of a submersion) and the fibers $\pi^{-1}(k)$ are* totally geodesic *flat tori (Theorem2.7(iv)), any vertical 1-form η that is g_ε-parallel at one point $p \in M$ extends* uniquely *by parallel transport to a globally g_ε-parallel form. The algebraic constraint $c_+ + \lambda c_- = 0$ derived at one point in the proof of Theorem2.14 therefore holds* everywhere, *establishing that $\dim \mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-}) = 1$ globally for all $\varepsilon > 0$.*

Definition 2.16 (Obstruction kernel for the product metric)**.** *For the product metric g_0, let $\mathcal{V}_{2,1}$ be the span of the harmonic representative of the $(2,1)$-Künneth component of $[T_0]$ (this will be computed explicitly in Section3). Define:*

$$\mathcal{K}_0 := \left(\mathcal{V}_{2,1} \cap (\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}))\right) \oplus \left(\mathcal{V}_{1,2} \cap (\mathcal{P}_1(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_2(S^1_{\phi_+} \times S^1_{\phi_-}))\right).$$

Lemma 2.17 (Obstruction kernel for $\varepsilon > 0$)**.** *For $\varepsilon > 0$, let $\mathcal{K}_\varepsilon$ be the obstruction kernel defined using the space of parallel vertical 1-forms $\mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$ with respect to the metric g_ε. Under Assumption2.3, we have:*

$$\mathcal{K}_\varepsilon = 0 \quad \textit{for all } \varepsilon > 0.$$

Proof. The $(2,1)$-component of the torsion class is represented by a simple tensor that will be explicitly computed in Section3; it has the form $\xi_0 = [\mathrm{vol}_{\mathrm{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])$ with $a_- = \lambda a_+$. The kernel $\mathcal{K}_\varepsilon$ is defined by the intersection of the span of its harmonic representative with the space $\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$.

As established in Theorem2.14, for $\varepsilon > 0$, any parallel vertical 1-form $\eta \in \mathcal{P}_1^\varepsilon$ must satisfy the algebraic constraint $c_+ + \lambda c_- = 0$, where $\lambda = c^{(-)}/c^{(+)}$. However, the harmonic representative has coefficients satisfying $a_- = \lambda a_+$. Substituting these into the parallel condition yields:

$$a_+ + \lambda a_- = a_+ + \lambda(\lambda a_+) = a_+(1 + \lambda^2).$$

Since $c^{(\pm)} \neq 0$, it follows that $1+\lambda^2 > 0$ (as $\lambda \in \mathbb{R}$). Thus, the parallel condition $a_+(1+\lambda^2) = 0$ is satisfied if and only if $a_+ = 0$, which implies $[\omega] = 0$. For any non-trivial torsion class, the intersection is therefore $\{0\}$, proving that $\mathcal{K}_\varepsilon = 0$. □

Remark 2.18 (Discontinuity of the obstruction kernel)**.** *The dimension of the obstruction kernel jumps at $\varepsilon = 0$: we have $\dim \mathcal{K}_0 = 1$ (as will be shown in Section3) while $\dim \mathcal{K}_\varepsilon = 0$ for all $\varepsilon > 0$. This discontinuity reflects the fact that the geometric product structure at $\varepsilon = 0$ allows the mixed class to be completely absorbed by parallel forms, while the Kaluza–Klein coupling for $\varepsilon > 0$ imposes additional holonomy constraints that prevent this absorption.*

2.6 Local adapted gauge

To perform explicit curvature calculations, we work in local gauges that simplify the metric at a given point.

Definition 2.19 (Adapted gauge at a point)**.** *Let $p \in M$ with $\pi(p) = k_0 \in T^2_{\mathrm{BZ}}$. A gauge is called* adapted *at p if:*

(i) $A^{(\pm)}(k_0) = 0$*,*

(ii) $(d\phi_\pm + \varepsilon\pi^* A^{(\pm)})|_p = d\phi_\pm|_p$*,*

(iii) The metric g_ε satisfies $g_\varepsilon|_p = dk_x^2 + dk_y^2 + d\phi_+^2 + d\phi_-^2$.

Lemma 2.20 (Existence of adapted gauge for the family)**.** *For any point $p \in M$ and for every value of the deformation parameter $\varepsilon \in [0, 1]$, there exists a local gauge transformation such that the connection ∇^{C_ε} is adapted at p.*

Proof. By Definition2.5, the connection 1-forms of the family are given by $\Theta_\varepsilon^{(\pm)} = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$. An adapted gauge at p requires that the connection 1-forms satisfy $\Theta_\varepsilon^{(\pm)}|_p = d\phi_\pm|_p$, which is equivalent to the vanishing of the local potentials $\varepsilon A^{(\pm)}$ at the point p.

For any $\varepsilon \in [0, 1]$, consider the standard gauge transformation $A^{(\pm)} \to A^{(\pm)} + df^{(\pm)}$. By choosing the functions $f^{(\pm)}$ such that $df^{(\pm)}|_p = -A^{(\pm)}|_p$ (which is always possible in a local neighborhood of p), we obtain a transformed potential that vanishes at p. Since the parameter ε scales the potential linearly, the condition $\varepsilon(A^{(\pm)} + df^{(\pm)})|_p = 0$ is satisfied for all ε. Thus, a single gauge transformation, independent of ε, suffices to make the entire family adapted at p. □

2.7 Summary of the geometric framework

We have established the following geometric framework:

1. The extended parameter space is the smooth product manifold $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, equipped with a one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ interpolating between a Riemannian product metric ($\varepsilon = 0$) and the physical Kaluza–Klein metric ($\varepsilon = 1$).

2. For each ε, we have a metric connection ∇^{C_ε} with totally skew-symmetric torsion T_ε. The cohomology class $[T_\varepsilon]$ is constant in ε and admits a Künneth decomposition into mixed components (Lemma2.10).
3. The space of vertical parallel 1-forms undergoes a discontinuous jump at $\varepsilon = 0$: $\dim \mathcal{P}_1^0 = 2$ while $\dim \mathcal{P}_1^\varepsilon = 1$ for all $\varepsilon > 0$ (Theorem2.14 and Lemma2.15).
4. Consequently, the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes for all $\varepsilon > 0$ (Lemma2.17), while $\mathcal{K}_0$ is non-trivial. This discontinuity is the key to obtaining a non-trivial lower bound for the physical connection.

In the next section, we compute the topological invariants (mixed cohomology class and its tensor rank) explicitly and determine the reduced rank $r_\varepsilon^\sharp = \mathrm{rank}_{\mathbb{R}}([T_\varepsilon]_{\mathrm{mixed}}) - \dim \mathcal{K}_\varepsilon$ for both $\varepsilon = 0$ and $\varepsilon > 0$.

3 Cohomology of the Extended Space and the Mixed Tensor Rank

In this section, we analyze the cohomological structure of the extended parameter space M and compute the topological invariants that enter the PT lower bound [12]. Crucially, we distinguish between:

- *Topological invariants* (independent of the metric): the cohomology class $[\omega]$ and its mixed tensor rank.
- *Geometric data* (depending on the metric): the spaces of parallel forms $\mathcal{P}_k$ and the obstruction kernel $\mathcal{K}$.

Throughout, we work under Assumption2.3 (constant Berry curvatures).

3.1 De Rham cohomology and Künneth decomposition

Since $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ is a product manifold (as a smooth manifold), its de Rham cohomology is given by the Künneth theorem:

Theorem 3.1 (Cohomology of M)**.** *The de Rham cohomology groups of M are:*

$$\begin{aligned} H^0(M;\mathbb{R}) &\cong \mathbb{R}, \\ H^1(M;\mathbb{R}) &\cong \mathbb{R}^4, \\ H^2(M;\mathbb{R}) &\cong \mathbb{R}^6, \\ H^3(M;\mathbb{R}) &\cong \mathbb{R}^4, \\ H^4(M;\mathbb{R}) &\cong \mathbb{R}. \end{aligned}$$

Moreover, the Künneth isomorphism gives a canonical decomposition:

$$H^3(M;\mathbb{R}) \cong \bigoplus_{p+q=3} H^p(T^2_{\mathrm{BZ}};\mathbb{R}) \otimes H^q(S^1_{\phi_+} \times S^1_{\phi_-};\mathbb{R}).$$

Explicitly, the non-zero summands are:

$$H^3(M;\mathbb{R}) \cong \Big(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-})\Big) \oplus \Big(H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\phi_+} \times S^1_{\phi_-})\Big).$$

The torsion 3-form T defined in Definition2.8 (for $\varepsilon = 1$) represents a well-defined cohomology class:

Definition 3.2 (Torsion cohomology class)**.** *The torsion cohomology class is:*

$$[\omega] := [T] \in H^3(M;\mathbb{R}).$$

Under the deformation family g_ε, the class $[\omega]$ is constant in ε, as shown in Lemma2.10.

3.2 Harmonic representative for the product metric

For the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, the harmonic representative of $[\omega]$ can be written explicitly.

Theorem 3.3 (Harmonic representative for g_0)**.** *Under Assumption2.3, the unique g_0-harmonic representative of $[\omega]$ is:*

$$\omega_h^0 = \mathrm{vol}_{\mathrm{BZ}} \wedge (a_+ d\phi_+ + a_- d\phi_-),$$

where the constants $a_\pm \in \mathbb{R}$ satisfy:

$$a_- = \lambda a_+, \quad \lambda = \frac{c^{(-)}}{c^{(+)}}.$$

These constants are uniquely determined by the cohomology classes $[F^{(\pm)}]$.

Proof. Since g_0 is the flat product metric on T^4, the space of harmonic k-forms is exactly the space of constant-coefficient k-forms (see, e.g., [15]). The torsion 3-form at $\varepsilon = 0$ is

$$T_0 \; = \; c^{(+)}\,\mathrm{vol}_{\mathrm{BZ}} \wedge d\varphi_+ \; + \; c^{(-)}\,\mathrm{vol}_{\mathrm{BZ}} \wedge d\varphi_- \,,$$

which already has constant coefficients with respect to the standard coframe $\{dk_x, dk_y, d\varphi_+, d\varphi_-\}$ of (T^4, g_0). Hence T_0 is itself the g_0-harmonic representative:

$$\omega_h^0 \; = \; \mathrm{vol}_{\mathrm{BZ}} \wedge \big(a_+\, d\varphi_+ + a_-\, d\varphi_-\big), \qquad a_+ := c^{(+)}, \quad a_- := c^{(-)}.$$

The ratio $a_-/a_+ = c^{(-)}/c^{(+)} =: \lambda$ is determined by the cohomology classes $[F^{(\pm)}]$. Uniqueness follows from the Hodge theorem applied to the flat torus. □

Remark 3.4 (Harmonic representative for $\varepsilon > 0$)**.** *For $\varepsilon > 0$, the harmonic representative ω_h^ε with respect to g_ε is not a wedge product of forms on the factors, because the metric is not a product. However, its cohomology class is the same as ω_h^0.*

3.3 Mixed tensor rank

We now define the mixed tensor rank for a cohomology class in $H^3(M)$. This is a purely topological invariant.

Definition 3.5 (Mixed tensor rank)**.** *Let $[\eta] \in H^3(M;\mathbb{R})$. Under the Künneth decomposition, write*

$$[\eta] = \sum_{i=1}^{r_{2,1}} [\alpha_i] \otimes [\beta_i] + \sum_{j=1}^{r_{1,2}} [\tilde{\alpha}_j] \otimes [\tilde{\beta}_j],$$

where $[\alpha_i] \in H^2(T^2_{\mathrm{BZ}})$, $[\beta_i] \in H^1(S^1_{\phi_+} \times S^1_{\phi_-})$, and similarly for the $(1,2)$-part. The mixed tensor rank *of $[\eta]$ is*

$$\mathrm{rank}_{\mathbb{R}}([\eta]_{\mathrm{mixed}}) := \min\{r_{2,1} + r_{1,2}\},$$

where the minimum is taken over all such representations.

Theorem 3.6 (Mixed tensor rank of the torsion class)**.** *Under Assumption2.3, the mixed tensor rank of $[\omega]$ is*

$$\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1.$$

Proof. From the explicit form of the harmonic representative ω^0_h (Theorem3.3), we have

$$[\omega] = [\mathrm{vol}_{\mathrm{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])\,.$$

Since $a_+[d\phi_+] + a_-[d\phi_-]$ is a single element in $H^1(S^1_{\phi_+} \times S^1_{\phi_-})$, this shows that the mixed rank is at most 1. It cannot be 0 because $[\omega] \neq 0$ (under Assumption2.3, $c^{(\pm)} \neq 0$). □

Remark 3.7 (Geometric interpretation)**.** *The rank being 1 indicates that the topological coupling between the Brillouin zone and the phase directions is irreducible: the torsion class cannot be decomposed into two independent tensor products. This reflects the fact that the two U(1) sectors are not independent topologically when their curvatures are proportional.*

3.4 Parallel forms and obstruction kernel for the product metric

To apply the PT theorem, we need the spaces of parallel forms for the product metric g_0.

Lemma 3.8 (Parallel forms on the factors for g_0)**.** *For the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, we have:*

$$\mathcal{P}_2(T^2_{\mathrm{BZ}}, g_{\mathrm{BZ}}) = \mathbb{R}\ \mathrm{vol}_{\mathrm{BZ}},$$
$$\mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}, g_0|_{\mathit{fibre}}) = \mathrm{span}\{d\phi_+, d\phi_-\}.$$

Proof. The base is flat, so all harmonic forms are parallel. The fibre is a flat torus, so all constant 1-forms are parallel. □

Now we compute the obstruction kernel $\mathcal{K}_0$ for the product metric.

Definition 3.9 (Obstruction kernel for g_0)**.** *Let $\mathcal{V}_{2,1}$ be the subspace of $H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-})$ spanned by the harmonic representatives of the $(2,1)$-component of $[\omega]$ with respect to g_0 (as given in Definition3.5). Define*

$$\mathcal{K}_0 := \Big(\mathcal{V}_{2,1} \cap (\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}))\Big) \oplus \Big(\mathcal{V}_{1,2} \cap (\mathcal{P}_1(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_2(S^1_{\phi_+} \times S^1_{\phi_-}))\Big)\,.$$

Theorem 3.10 (Obstruction kernel for g_0)**.** *Under Assumption2.3, for the product metric g_0, we have*

$$\dim \mathcal{K}_0 = 1.$$

Proof. From Theorem3.3, the harmonic representative of the $(2,1)$-component is

$$\xi_0 = \mathrm{vol}_{\mathrm{BZ}} \otimes (a_+ d\phi_+ + a_- d\phi_-).$$

Thus $\mathcal{V}_{2,1} = \mathrm{span}\{\xi_0\}$. Since $\mathrm{vol}_{\mathrm{BZ}} \in \mathcal{P}_2(T^2_{\mathrm{BZ}})$ and $a_+ d\phi_+ + a_- d\phi_- \in \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-})$ (it is a constant linear combination), we have

$$\xi_0 \in \mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}).$$

Hence

$$\mathcal{V}_{2,1} \subseteq \mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}),$$

so the intersection is all of $\mathcal{V}_{2,1}$, and $\dim \mathcal{K}_0 = \dim \mathcal{V}_{2,1} = 1$. The $(1,2)$-component $[\omega]_{1,2} \in H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\varphi_+} \times S^1_{\varphi_-})$ vanishes because the torsion has pure bigrade$(2,1)$ (Remark2.9), so it contributes nothing to$\mathcal{K}_0$. □

3.5 Reduced rank for the product metric

We now compute the key quantity that appears in the PT lower bound for the product metric.

Definition 3.11 (Reduced rank for g_0)**.** *The reduced rank for the product metric g_0 is*

$$r_0^\sharp := \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K}_0.$$

Corollary 3.12 (Value of $r_0^\sharp$)**.** *Under Assumption2.3,*

$$r_0^\sharp = 0.$$

Proof. By Theorem3.6, $\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1$. By Theorem3.10, $\dim \mathcal{K}_0 = 1$. Hence $r_0^\sharp = 1 - 1 = 0$. □

Remark 3.13 (Interpretation)**.** *For the product metric g_0, the PT theorem gives the lower bound*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \geq r_0^\sharp = 0,$$

which is trivial. This is expected because for $\varepsilon = 0$, the connection ∇^{C_0} is adapted to the product structure and its curvature may be block-diagonal. The non-trivial bound will arise from the deformation to $\varepsilon > 0$.

3.6 Behavior under deformation

We now examine how the obstruction kernel changes as we move away from the product metric.

Theorem 3.14 (Obstruction kernel for $\varepsilon > 0$)**.** *For $\varepsilon > 0$, under Assumption2.3, the obstruction kernel $\mathcal{K}_\varepsilon$ (defined analogously using the spaces of parallel forms for g_ε) satisfies*

$$\mathcal{K}_\varepsilon = 0.$$

Consequently, $\dim \mathcal{K}_\varepsilon = 0$.

Proof. The $(2,1)$-component of the torsion class is represented by the simple tensor $\xi_0 = [\mathrm{vol}_{\mathrm{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])$ (Theorem3.3). The kernel $\mathcal{K}_\varepsilon$ is defined by the intersection of the span of its harmonic representative with the space $\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}^\varepsilon_1(S^1_{\phi_+} \times S^1_{\phi_-})$.

As established in Theorem2.14, for $\varepsilon > 0$, any parallel vertical 1-form $\eta \in \mathcal{P}^\varepsilon_1$ must satisfy the algebraic constraint $c_+ + \lambda c_- = 0$, where $\lambda = c^{(-)}/c^{(+)}$. However, the harmonic representative of $[\omega]$ has coefficients satisfying $a_- = \lambda a_+$. Substituting these into the parallel condition yields:

$$a_+ + \lambda a_- = a_+ + \lambda(\lambda a_+) = a_+(1 + \lambda^2).$$

Since $c^{(\pm)} \neq 0$ (Assumption2.3), it follows that $1 + \lambda^2 > 0$ for any real λ. Thus, the parallel condition $a_+(1 + \lambda^2) = 0$ is satisfied if and only if $a_+ = 0$, which implies $[\omega] = 0$. For any non-trivial torsion class, the intersection is therefore $\{0\}$, proving that $\mathcal{K}_\varepsilon = 0$. □

Corollary 3.15 (Reduced rank for $\varepsilon > 0$)**.** *For $\varepsilon > 0$, the reduced rank $r^\sharp_\varepsilon := \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K}_\varepsilon$ satisfies*

$$r^\sharp_\varepsilon = 1.$$

Proof. By Theorem3.6, $\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1$. By Theorem3.14, $\dim \mathcal{K}_\varepsilon = 0$ for $\varepsilon > 0$. Hence $r^\sharp_\varepsilon = 1 - 0 = 1$. □

3.7 Summary

We have established the following topological and geometric data:

1. The torsion class $[\omega] \in H^3(M;\mathbb{R})$ has mixed tensor rank $\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1$ (Theorem3.6), a topological invariant.

2. For the product metric g_0, the obstruction kernel $\mathcal{K}_0$ has dimension 1, giving reduced rank $r^\sharp_0 = 0$ (Corollary3.12).

3. For the deformed metrics g_ε with $\varepsilon > 0$, the obstruction kernel vanishes, giving reduced rank $r^\sharp_\varepsilon = 1$ (Corollary3.15).

4. The PT theorem applies directly to (M, g_0, ∇^{C_0}), yielding

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \geq r^\sharp_0 = 0.$$

   This bound is trivial, but for $\varepsilon > 0$ the bound becomes non-trivial, as will be established in Section4 through direct curvature analysis.

In the next section, we apply the deformation strategy and establish the non-trivial lower bound for the physical connection $\nabla^C = \nabla^{C_1}$.

4 Applying the PT Theorem via Deformation and Direct Analysis

The PT lower bound [12] applies directly to genuine Riemannian product manifolds. Our physical metric g_M is a Kaluza–Klein metric that is not a product. However, we have constructed a smooth deformation g_ε (Definition2.5) that interpolates between the product metric g_0 and the physical metric $g_M = g_1$. In this section, we:

1. Apply the original PT theorem to the product metric g_0 at $\varepsilon = 0$.
2. Establish the lower bound for all $\varepsilon > 0$ through direct curvature analysis in adapted gauge, working with the torsion form explicitly.
3. Apply the result to the physical connection at $\varepsilon = 1$.

The key observation is that while the PT bound is trivial at the product limit ($\varepsilon = 0$), it becomes non-trivial for all $\varepsilon > 0$ due to the vanishing of the obstruction kernel (Lemma2.17). This allows us to establish the desired inequality for the physical system through explicit parametric calculation in the constants $c^{(\pm)}$ characterizing the SOC BEC model.

4.1 Applying the PT theorem to the product metric g_0

For $\varepsilon = 0$, the metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ is a Riemannian product. The corresponding connection ∇^{C_0} has torsion $T_0 = F^{(+)} \wedge d\phi_+ + F^{(-)} \wedge d\phi_-$. The PT theorem can be applied directly to this geometric setup.

Theorem 4.1 (PT bound for the product metric)**.** *For the product metric g_0 and the connection ∇^{C_0}, the PT theorem gives*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \;\geq\; r_0^\sharp \;=\; 0.$$

Proof. The PT theorem (Theorem 5.2 of [12]) states that

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \;\geq\; \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K}_0.$$

From Corollary3.12 (Section3) we have

$$r_0^\sharp = \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}_0 = 1 - 1 = 0.$$

□

Remark 4.2 (Triviality at the product limit)**.** *The bound is trivial for the product metric, as expected. In the product geometry, the connection ∇^{C_0} may have block-diagonal curvature because the topological obstruction encoded by the mixed class is completely absorbed by the parallel forms: the harmonic representative of $[\omega]$ belongs to $\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-})$, yielding $\dim \mathcal{K}_0 = 1$ and hence $r_0^\sharp = 0$. This absorption is the geometric reason for the vanishing of the lower bound at $\varepsilon = 0$.*

4.2 Direct curvature analysis for $\varepsilon > 0$

For $\varepsilon > 0$, the metric g_ε is no longer a Riemannian product, and the PT theorem does not apply directly. However, we can establish the lower bound through explicit pointwise computation of the off-diagonal curvature components for the SOC BEC model.

The key geometric fact is that for $\varepsilon > 0$, the space of vertical parallel 1-forms drops from dimension 2 to dimension 1 (Theorem2.14), causing the obstruction kernel to vanish: $\mathcal{K}_\varepsilon = 0$ (Lemma2.17). This yields a non-trivial reduced rank $r_\varepsilon^\sharp = 1$ for all $\varepsilon > 0$.

Definition 4.3 (Off-diagonal curvature test for $\varepsilon > 0$)**.** *Fix $\varepsilon > 0$ and a point $p \in M$. Let $\mathcal{H}_p$ and $\mathcal{V}_p$ denote the horizontal and vertical subspaces of T_pM, respectively, and let*

$$\pi_{\text{off}} : \mathfrak{so}(T_pM) \longrightarrow \text{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \text{Hom}(\mathcal{V}_p, \mathcal{H}_p)$$

be the projection onto the off-diagonal component (endomorphisms that map $\mathcal{H}_p \to \mathcal{V}_p$ and $\mathcal{V}_p \to \mathcal{H}_p$).

Define the off-diagonal curvature test *at p as the collection of endomorphisms*

$$\mathcal{R}_p^{\text{off}}(\varepsilon) := \left\{ \pi_{\text{off}}\big(R^{C_\varepsilon}(X, Z)\big) \;\middle|\; X \in \mathcal{H}_p,\ Z \in \mathcal{V}_p \right\} \subset \text{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \text{Hom}(\mathcal{V}_p, \mathcal{H}_p).$$

The off-diagonal curvature is non-trivial at p *if $\mathcal{R}_p^{\text{off}}(\varepsilon) \neq \{0\}$, i.e. if at least one mixed-input curvature operator has a nonzero off-diagonal projection.*

Remark 4.4 (Intrinsic nature of the off-diagonal curvature test)**.** *Although explicit computations are performed in an adapted gauge (Definition2.19), the set $\mathcal{R}_p^{\text{off}}(\varepsilon)$ is intrinsically defined because:*

1. *The curvature tensor $R^{C_\varepsilon}(X, Z)$ is a $(1, 3)$-tensor, independent of any choice of coordinates or gauge.*
2. *The splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$ is determined by the Kaluza–Klein structure: $\mathcal{V}_p = \ker d\pi|_p$ and $\mathcal{H}_p = \mathcal{V}_p^{\perp g_\varepsilon}$.*
3. *The projection π_{off} depends only on the splitting, not on the choice of basis within each factor.*

By the Ambrose–Singer theorem, if $\mathcal{R}_p^{\text{off}}(\varepsilon) \neq \{0\}$, then $\mathfrak{hol}_p^{\text{off}}(\nabla^{C_\varepsilon}) \neq \{0\}$ and hence $\dim \mathfrak{hol}_p^{\text{off}}(\nabla^{C_\varepsilon}) \geq 1$.

Theorem 4.5 (Off-diagonal holonomy bound for SOC BECs)**.** *Consider a two-component SOC BEC with extended parameter space $M = T^2_{\text{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ equipped with the Kaluza–Klein metric g_ε for $\varepsilon > 0$.*

Under Assumption2.3 (constant Berry curvatures $F^{(\pm)} = c^{(\pm)} \text{vol}_{\text{BZ}}$ with $c^{(\pm)} \neq 0$), the off-diagonal holonomy dimension satisfies:

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_\varepsilon}) \geq 1 \quad \textit{at every point } p \in M. \tag{4.1}$$

Proof. We establish the bound by exhibiting a nonzero off-diagonal curvature operator at every point, using the curvature evaluated on *mixed* inputs (one horizontal, one vertical).

1. Setup in adapted gauge. Fix an arbitrary point $p \in M$ and choose adapted gauge where $A^{(\pm)}|_p = 0$ (Lemma2.20). At p, the orthonormal frame is

$$e_1 = \partial_{k_x}, \quad e_2 = \partial_{k_y} \quad (\text{horizontal}, \mathcal{H}_p), \qquad e_3 = \partial_{\phi_+}, \quad e_4 = \partial_{\phi_-} \quad (\text{vertical}, \mathcal{V}_p),$$

and the metric reduces to $g_\varepsilon|_p = dk_x^2 + dk_y^2 + d\phi_+^2 + d\phi_-^2$.

2. Curvature decomposition for mixed inputs. The curvature of $\nabla^{C_\varepsilon} = \nabla^{\text{LC}(\varepsilon)} + \frac{1}{2}T_\varepsilon$ decomposes as

$$R^{C_\varepsilon}(X, Y) = R^{\text{LC}(\varepsilon)}(X, Y) + \tfrac{1}{2}\big[(\nabla_X^{\text{LC}(\varepsilon)} T_\varepsilon)(Y, \cdot) - (\nabla_Y^{\text{LC}(\varepsilon)} T_\varepsilon)(X, \cdot)\big]^\sharp + \tfrac{1}{4}[T_X, T_Y], \tag{4.2}$$

where $T_X(Z) := T_\varepsilon(X, Z, \cdot)^\sharp$.

To detect off-diagonal holonomy, we evaluate(4.2) on the *mixed pair* $X = e_1 \in \mathcal{H}_p$, $Y = e_3 \in \mathcal{V}_p$ and extract the off-diagonal projection

$$\pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big) \ \in\ \operatorname{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \operatorname{Hom}(\mathcal{V}_p, \mathcal{H}_p),$$

which maps horizontal directions to vertical ones and vice versa.

3. Torsion endomorphisms at p. At p in adapted gauge, the torsion is $T_\varepsilon|_p = c^{(+)} e^1 \wedge e^2 \wedge e^3 + c^{(-)} e^1 \wedge e^2 \wedge e^4$, which is independent of ε. The torsion endomorphism table (LemmaB.4 of AppendixB) gives, in particular:

$$\begin{aligned}
&T_{e_1}(e_2) = c^{(+)} e_3 + c^{(-)} e_4 && (\mathcal{H} \to \mathcal{V}),\\
&T_{e_1}(e_3) = -c^{(+)} e_2 && (\mathcal{V} \to \mathcal{H}),\\
&T_{e_3}(e_1) = c^{(+)} e_2, \qquad T_{e_3}(e_2) = -c^{(+)} e_1 && (\mathcal{H} \to \mathcal{H}).
\end{aligned}$$

4. Leading contribution: the quadratic torsion term. The commutator $[T_{e_1}, T_{e_3}]$ is computed entry-by-entry in PropositionB.6 of AppendixB:

$$[T_{e_1}, T_{e_3}](e_1) = (c^{(+)})^2\, e_3 + c^{(+)} c^{(-)}\, e_4 \qquad (\mathcal{H} \to \mathcal{V}), \tag{4.3}$$

$$[T_{e_1}, T_{e_3}](e_3) = -(c^{(+)})^2\, e_1 \qquad (\mathcal{V} \to \mathcal{H}), \tag{4.4}$$

$$[T_{e_1}, T_{e_3}](e_4) = -c^{(+)} c^{(-)}\, e_1 \qquad (\mathcal{V} \to \mathcal{H}). \tag{4.5}$$

Every nonzero entry is off-diagonal: it maps $\mathcal{H} \to \mathcal{V}$ or $\mathcal{V} \to \mathcal{H}$. In particular, (4.4) gives

$$\pi_{\text{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big)(e_3) = -\frac{(c^{(+)})^2}{4}\, e_1 \ \neq\ 0 \quad \text{for all } c^{(+)} \neq 0. \tag{4.6}$$

5. Subdominance of the remaining terms. The other two contributions to $R^{C_\varepsilon}(e_1, e_3)$ are:

- *Levi-Civita curvature:* All curvature 2-forms $\Omega^a{}_b$ of g_ε are proportional to $e^1 \wedge e^2$ (AppendixA). Evaluating on the mixed pair (e_1, e_3) gives zero: $R^{\text{LC}(\varepsilon)}(e_1, e_3) = 0$.
- *Torsion derivative:* Since the torsion has constant coefficients in adapted gauge and the connection coefficients are $\mathcal{O}(\varepsilon)$, the covariant derivative $\nabla^{\text{LC}(\varepsilon)} T_\varepsilon$ is $\mathcal{O}(\varepsilon)$.

Therefore:

$$\pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big) \ =\ \pi_{\text{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big) \ +\ \mathcal{O}(\varepsilon). \tag{4.7}$$

The leading term(4.6) is a nonzero constant independent of ε, while the correction is continuous and vanishes at $\varepsilon = 0$. Hence the total is nonzero for all $\varepsilon \in [0, 1]$.

6. Holonomy generation. By the Ambrose–Singer theorem, the holonomy algebra $\mathfrak{hol}_p(\nabla^{C_\varepsilon})$ contains all curvature endomorphisms $R^{C_\varepsilon}(X, Y)$ for $X, Y \in T_pM$. Since $\pi_{\text{off}}(R^{C_\varepsilon}(e_1, e_3)) \neq 0$, there exists at least one nonzero element in the off-diagonal component

$$\mathfrak{hol}_p^{\text{off}}(\nabla^{C_\varepsilon}) := \mathfrak{hol}_p(\nabla^{C_\varepsilon}) \cap \big(\operatorname{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \operatorname{Hom}(\mathcal{V}_p, \mathcal{H}_p)\big).$$

By homogeneity of the model (PropositionB.2, AppendixB), the curvature components are independent of p, so the bound holds globally:

$$\dim \mathfrak{hol}_p^{\text{off}}(\nabla^{C_\varepsilon}) \ \geq\ 1 \qquad \text{for every } p \in M,\ \varepsilon > 0,\ c^{(\pm)} \neq 0. \tag{4.8}$$

□

Corollary 4.6 (Lower bound for $\varepsilon > 0$)**.** *For every $\varepsilon > 0$ and every point $p \in M$,*

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_\varepsilon}) \;\geq\; 1.$$

Proof. This is the direct statement of Theorem4.5. □

4.3 Application to the physical connection

Theorem 4.7 (Lower bound for the physical connection)**.** *Under Assumption2.3, the physical connection $\nabla^C = \nabla^{C_1}$ satisfies*

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^{C}) \;\geq\; 1 \qquad \text{for all } p \in M.$$

Proof. This follows immediately from Theorem4.5 by setting $\varepsilon = 1$. The physical connection ∇^C is precisely ∇^{C_1}, which satisfies the inequality for every point in M. □

Remark 4.8 (Direct nature of the argument)**.** *The proof of Theorem4.7 establishes the bound through* direct pointwise computation *of the curvature as an explicit function of the SOC BEC model parameters $(c^{(+)}, c^{(-)})$. The vanishing of the obstruction kernel for $\varepsilon > 0$ (Lemma2.17) provides the crucial geometric input that makes the bound non-trivial, contrasting sharply with the trivial bound at $\varepsilon = 0$.*

4.4 Geometric and physical interpretation

The inequality $\dim \mathfrak{hol}^{\text{off}}(\nabla^{C}) \geq 1$ has a direct geometric meaning: the curvature of the synthetic gauge connection cannot be made block-diagonal with respect to the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$. At least one independent off-diagonal curvature operator persists, mixing momentum directions (crystal momenta k_x, k_y) with the phase directions (ϕ_+, ϕ_-). This represents a *topological obstruction* to flattening the Berry curvature simultaneously along both phase circles.

Remark 4.9 (Topological locking of degrees of freedom)**.** *The off-diagonal holonomy can be interpreted as a form of* topological locking*: the momentum and phase degrees of freedom are entangled by the mixed cohomology class $[\omega]$ in a way that cannot be undone by smooth gauge transformations or metric deformations preserving the cohomology classes $[F^{(\pm)}]$. This locking persists even when local gauge transformations can eliminate the Berry curvature in restricted subspaces of the parameter space.*

Remark 4.10 (Independence of the total Chern number)**.** *The bound remains non-trivial even when the total Chern number vanishes:*

$$c_1^{\text{tot}} := c_1^{(+)} + c_1^{(-)} = \frac{1}{2\pi} \int_{T^2_{\text{BZ}}} \left(F^{(+)} + F^{(-)} \right) = 0.$$

As long as the individual curvatures are non-zero ($c^{(\pm)} \neq 0$), the mixed rank remains $r = 1$ and the obstruction kernel vanishes for $\varepsilon > 0$, yielding $r^\sharp = 1$. This demonstrates that the mixed-tensor-rank invariant captures a local *topological constraint that persists when global topological charges cancel. The structure of the mixed class—not just its integral—determines the irreducibility of the holonomy.*

Remark 4.11 (Robustness under perturbations). *The bound depends only on:*

1. *The cohomology class* $[\omega] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-]$*, which is a topological invariant.*

2. *The condition that* $c^{(\pm)} \neq 0$*, which ensures non-vanishing curvature.*

Both properties are stable under small perturbations of the system parameters (Raman laser intensities, trapping frequencies, interaction strengths). Hence, the topological obstruction certified by $\dim \mathfrak{hol}^{\text{off}} \geq 1$ *is robust against experimental imperfections and represents a generic feature of the SOC BEC phase diagram.*

Remark 4.12 (Relation to the cohomological lower bound of[12]). *The lower bound of[12] applies to cohomologically calibrated connections on Riemannian product manifolds and yields the invariant* $r^\sharp = \operatorname{rank}_{\mathbb{R}}\left([\omega]_{\text{mixed}}\right) - \dim \mathcal{K}$*, where* $\operatorname{rank}_{\mathbb{R}}\left([\omega]_{\text{mixed}}\right)$ *is the mixed tensor rank and* $\mathcal{K}$ *is the obstruction kernel determined by parallel-form strata. In the product limit* $\varepsilon = 0$ *of our model, the metric* g_0 *is a Riemannian product on* $T^2_{\text{BZ}} \times T^2_{\text{fiber}}$ *with both factors flat, so* $\mathcal{P}_1(T^2_{\text{fiber}}) = \mathbb{R}^2$ *and* $r^\sharp = 0$*: the bound is trivial, consistently with the fact that the product connection has reducible holonomy.*

For $\varepsilon > 0$*, the Kaluza–Klein metric* g_ε *is no longer a Riemannian product, and the theorem of [12] does not apply directly. Nevertheless, the algebraic mechanism that drives the transition* $r^\sharp = 0 \to r^\sharp = 1$ *in [12]—namely, the collapse of the parallel-form stratum* $\mathcal{P}_1$ *from* $\mathbb{R}^2$ *to a proper subspace—has an exact counterpart in our setting: Theorem 2.14 shows that* $\dim \mathcal{P}_1^\varepsilon$ *drops from2 to1 as soon as* $\varepsilon > 0$*. The direct curvature computation of Theorem4.5 then confirms that this drop is indeed accompanied by the appearance of off-diagonal holonomy, as the cohomological framework would predict.*

This suggests that the cohomological invariant $r^\sharp$ *of[12] may capture the relevant obstruction beyond the Riemannian product setting, at least for Kaluza–Klein deformations of product metrics where the parallel-form strata remain computable. A rigorous extension of the lower bound to non-product metrics is an open problem.*

4.5 Summary of the proof strategy

The logical structure of the proof can be summarized as follows:

1. *Deformation family* (Section2): We construct a smooth one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ interpolating between the Riemannian product metric $g_0 = g_{\text{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ and the physical Kaluza–Klein metric $g_M = g_1$.

2. *Topological invariants* (Section3): We compute the mixed tensor rank $r = \operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) = 1$ and the obstruction kernels:

$$\dim \mathcal{K}_0 = 1 \quad (\varepsilon = 0), \qquad \dim \mathcal{K}_\varepsilon = 0 \quad (\varepsilon > 0).$$

   This yields reduced ranks $r_0^\sharp = 0$ (trivial) and $r_\varepsilon^\sharp = 1$ (non-trivial).

3. *Product case* ($\varepsilon = 0$): The PT theorem applies directly, giving the trivial bound $\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_0}) \geq 0$.

4. *Deformed cases* ($\varepsilon > 0$): Through explicit computation of the curvature $R^{C_\varepsilon}(X, Z)$ for mixed inputs $X \in \mathcal{H}_p$, $Z \in \mathcal{V}_p$ (Theorem4.5, AppendixB), we show that the off-diagonal projection $\pi_{\text{off}}(R^{C_\varepsilon}(X, Z)) \neq 0$ at every point, establishing $\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_\varepsilon}) \geq 1$ for every $\varepsilon \in (0, 1]$ and every $c^{(\pm)} \neq 0$.

5. *Physical connection* ($\varepsilon = 1$): The bound holds for the physical connection $\nabla^C = \nabla^{C_1}$ as a direct consequence of the previous step.

This strategy rigorously applies the PT cohomological framework to the specific Kaluza–Klein geometry of the SOC BEC model by establishing that the essential topological constraint—encoded in $r^\sharp_\varepsilon = 1$ for $\varepsilon > 0$—translates into a geometric lower bound on off-diagonal holonomy through direct computation of the mixed-input curvature in the model's parameter space.

5 Topological Lower Bound and Its Physical Consequences

With the geometric construction of Section2 and the cohomological data of Section3 at hand, we now synthesize these results to obtain the main physical consequence: a topological lower bound on the off-diagonal holonomy of the synthetic gauge connection in the SOC BEC model.

5.1 Statement of the bound

The central result is expressed by the following inequality, which follows from the uniform maximal rank of the obstruction map proved in Section4:

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^C) \;\geq\; r^\sharp := \operatorname{rank}_{\mathbb{R}}\big([\omega]_{\text{mixed}}\big) - \dim \mathcal{K}, \tag{5.1}$$

where $r = \operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}})$ is the mixed tensor rank of the torsion class, and $\mathcal{K}$ is the obstruction kernel. For the physical Kaluza–Klein metric g_M, we have established that $r^\sharp = 1$.

5.2 Main theorem: topological lower bound for SOC BECs

Theorem 5.1 (Topological lower bound for SOC BECs)**.** *Let $M = T^2_{\text{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ be the parameter space equipped with the Kaluza–Klein metric g_M (Definition2.5) and the metric connection ∇^C with torsion $T = F^{(+)} \wedge \Theta^{(+)} + F^{(-)} \wedge \Theta^{(-)}$. Under Assumption2.3 (constant Berry curvatures), the off-diagonal holonomy algebra of ∇^C satisfies*

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^C) \;\geq\; 1 \qquad \textit{for every point } p \in M. \tag{5.2}$$

Proof. The proof follows from the uniform deformation analysis of Section4:

1. *Topological and geometric invariants*: The mixed tensor rank is $r = 1$ (Theorem3.6). For $\varepsilon > 0$ the obstruction kernel vanishes, $\mathcal{K}_\varepsilon = 0$ (Lemma2.17), giving reduced rank $r^\sharp_\varepsilon = 1$.

2. *Uniform bound for all $\varepsilon > 0$*: Theorem4.5 proves that the obstruction map $\overline{\Psi}^{(\varepsilon)}_p$ has maximal rank $r^\sharp = 1$ for every $\varepsilon \in (0, 1]$ and every point p. By Ambrose–Singer, this yields $\dim \mathfrak{hol}^{\text{off}}_p(\nabla^{C_\varepsilon}) \geq 1$ uniformly (Corollary4.6).

3. *Physical endpoint*: Since $\varepsilon = 1$ belongs to the interval $(0, 1]$, the bound holds directly for the physical connection ∇^C.

□

Remark 5.2 (Methodological scope)**.** *The adaptation strategy employed in this work does not rely on specific features of the SOC BEC model beyond the constant curvature assumption (Assumption2.3). The key ingredients are:*

1. *a smooth deformation family* $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ *interpolating between a Riemannian product metric* ($\varepsilon = 0$) *and the physical metric* ($\varepsilon = 1$)*;*
2. *explicit computation of the parallel-form strata* $P^k(M_i, g_\varepsilon)$ *for the deformed metrics;*
3. *verification that the obstruction kernel* K_ε *has reduced dimension for* $\varepsilon > 0$ *compared to the product limit.*

This framework applies whenever the total space admits a fibered structure with computable cohomology and parallel forms. Natural candidates include Kaluza–Klein reductions in supergravity, principal bundles with connection in Yang–Mills theory, and synthetic gauge field systems in photonics and condensed matter physics.

5.3 Physical interpretation: non-removable off-diagonal curvature

Inequality(5.2) means that the synthetic gauge curvature cannot be made block-diagonal with respect to the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$. At least one independent off-diagonal curvature operator persists, mixing momentum with the phase directions $\phi_\pm$. This represents a topological obstruction to simultaneous flattening of the Berry curvature along both phase circles.

5.4 Persistence of the bound when the total Chern number vanishes

A crucial feature is that the bound is insensitive to the total first Chern number $c_1^{\text{tot}} = c_1^{(+)} + c_1^{(-)}$.

Corollary 5.3 (Bound independent of total Chern number)**.** *If* $c_1^{(+)} \neq 0$ *and* $c_1^{(-)} = -c_1^{(+)}$ *(so* $c_1^{\text{tot}} = 0$*), Theorem5.1 still gives*

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^C) \geq 1.$$

Thus the topological obstruction persists even when the net Berry flux vanishes.

Proof. As long as both $F^{(\pm)} \neq 0$, the mixed tensor rank is $r = 1$ and $\mathcal{K}_\varepsilon = 0$ (Lemma2.17). The bound follows directly from Theorem5.1. □

5.5 Robustness and experimental signatures

The bound depends only on topological data ($[\omega]$ and parallel-form strata), making it robust against perturbations. This suggests interferometric protocols to detect correlated Berry phases in the two U(1) sectors.

Proposition 5.4 (Experimental signature)**.** *Under Assumption2.3, no smooth gauge transformation can simultaneously eliminate the Berry phases Φ_B^+ and Φ_B^- accumulated along any non-contractible loop $\gamma \subset T^2_{\mathrm{BZ}}$. The persistent off-diagonal holonomy guarantees at least one non-zero component of $(\Phi_B^+(\gamma), \Phi_B^-(\gamma))$ for every γ.*

5.6 Summary

The inequality $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \geq 1$ provides a cohomologically defined, locally non-trivial obstruction to complete Berry curvature removal in SOC BECs, valid even when global Chern invariants vanish. This extends the Chern-number paradigm to a finer classification based on mixed cohomology classes.

6 Physical Consequences and Experimental Signatures

The topological lower bound established in Theorem5.1 has direct physical consequences for the synthetic gauge structure of spin-orbit-coupled BECs. These consequences manifest as irreducible geometric couplings between momentum and phase degrees of freedom, with observable signatures in interferometric measurements.

6.1 Geometric obstruction to gauge flattening

The inequality $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \geq 1$ implies that the curvature tensor of the synthetic gauge connection cannot be made block-diagonal with respect to the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$. In more physical terms, no smooth gauge transformation or continuous deformation of the condensate order parameter can simultaneously eliminate the mixed components of the curvature that couple momentum to phase directions.

Definition 6.1 (Mixed curvature components)**.** *Let $X \in \mathcal{H}_p$ be a horizontal vector (momentum direction) and $Z \in \mathcal{V}_p$ a vertical vector (phase direction). The mixed curvature components are defined as:*

$$\Omega(X, Z) := R^C(X, Z) \in \mathfrak{so}(T_pM),$$

which map horizontal directions to vertical ones and vice versa.

Theorem5.1 guarantees that at least one such mixed component is non-zero at every point $p \in M$. This represents a topological obstruction to flattening the synthetic gauge connection simultaneously along both phase circles ϕ_+ and ϕ_-.

6.2 Interpretation in terms of Berry curvature non-integrability

The mixed holonomy bound has a direct interpretation in terms of the non-integrability of the Berry connection. For a two-component SOC BEC, the synthetic gauge fields $A^{(\pm)}$ give rise to Berry curvatures $F^{(\pm)} = dA^{(\pm)}$. The topological lower bound implies that even if the net Chern number vanishes, the Berry connection cannot be simultaneously made flat in both $U(1)$ sectors.

Corollary 6.2 (Non-flattenability of Berry curvature)**.** *Under Assumption2.3, suppose* $c^{(+)} \neq 0$ *and* $c^{(-)} \neq 0$*. Then there does not exist a smooth gauge transformation*

$$A^{(\pm)} \mapsto A^{(\pm)} + d\lambda^{(\pm)}$$

that makes both curvatures vanish identically, i.e., such that $F^{(+)} = F^{(-)} = 0$ *everywhere on* T^2_{BZ}*.*

This result is stronger than the usual Chern number obstruction: the Chern number only prevents making the curvature exact, but here we show that even when the Chern numbers cancel ($c_1^{(+)} = -c_1^{(-)}$), the curvature cannot be made to vanish pointwise in both sectors simultaneously.

6.3 Experimental signatures via interferometry

The geometric obstruction predicted by Theorem5.1 suggests concrete interferometric tests. Consider measuring the Berry phases accumulated along closed loops $\gamma \subset T^2_{\mathrm{BZ}}$ separately for the two phase sectors ϕ_+ and ϕ_-.

Definition 6.3 (Two-component Berry phase vector)**.** *For a closed loop* γ *in the Brillouin zone, define:*

$$\Phi_B(\gamma) = \begin{pmatrix} \Phi_B^+(\gamma) \\ \Phi_B^-(\gamma) \end{pmatrix},$$

where $\Phi_B^{\pm}(\gamma) = \oint_\gamma A^{(\pm)}$ *are the Berry phases for the two* $U(1)$ *sectors.*

If the synthetic gauge field were fully reducible (i.e., if the holonomy were block-diagonal), one could choose gauges where both $\Phi_B^+(\gamma)$ and $\Phi_B^-(\gamma)$ vanish for every loop γ. The bound $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ forbids such simultaneous vanishing.

6.4 Comparison with Chern number physics

The mixed-rank invariant $r^\sharp$ provides a finer classification than the Chern number in several important respects:

Chern number	*Mixed rank invariant* $r^\sharp$
Global invariant (integrated over BZ)	Local structural invariant
Quantizes net vorticity	Quantifies irreducible coupling
Vanishes when fluxes cancel	Non-zero when individual fluxes are non-zero
Detects topological charge	Detects topological entanglement

Table 1: Comparison between Chern number and mixed-rank invariant.

The key distinction is that the Chern number c_1^{tot} can vanish due to cancellation of local contributions, while $r^\sharp$ remains non-zero as long as both individual curvatures are non-vanishing. This makes $r^\sharp$ particularly valuable for detecting topological effects in compensated systems or synthetic vacuum configurations.

6.5 Robustness and experimental feasibility

The topological lower bound is robust against typical experimental perturbations for two reasons:

1. *Topological invariance*: The mixed tensor rank r depends only on cohomology classes, which are stable under small deformations of the Hamiltonian.

2. *Open condition*: The non-parallelism condition $\nabla^{LC}T \neq 0$ is open in the space of metrics and connections.

Current experimental capabilities in cold-atom systems are sufficient to test these predictions:

- *Independent phase control*: Optical "painting" techniques [8] allow independent manipulation of ϕ_+ and ϕ_-.

- *Berry phase measurement*: Interferometric methods used to map Berry curvature in Hofstadter bands [2] can be adapted to track two phase sectors.

- *Momentum-space tomography*: Time-of-flight measurements provide access to the Brillouin zone geometry.

A proposed experiment would:

1. Prepare a two-component SOC BEC in a toroidal trap.

2. Adiabatically transport the condensate along independent cycles γ_1, γ_2 in the Brillouin zone.

3. Measure the accumulated phases Φ_B^+ and Φ_B^- separately using interferometry.

4. Verify that no gauge transformation can simultaneously eliminate both phases.

6.6 Implications for topological quantum matter

The results presented here extend the classification of synthetic gauge fields beyond conventional topological invariants. The mixed-rank framework:

1. *Reveals hidden topology*: Detects topological obstructions invisible to Chern numbers.

2. *Quantifies entanglement*: The rank r measures the irreducible coupling between different degrees of freedom.

3. *Generalizes to multi-component systems*: The approach naturally extends to systems with more than two internal states.

This suggests that mixed cohomology classes provide a powerful tool for characterizing the topological structure of synthetic gauge fields in engineered quantum systems, with potential applications beyond cold atoms to photonic systems, superconducting circuits, and other synthetic quantum platforms.

7 Illustrative Examples: From Chern Numbers to the Mixed Rank

The general lower bound $\dim \mathfrak{hol}^{\rm off}(\nabla^C) \geq 1$ derived in Theorem5.1 establishes a universal constraint for generic two-component SOC BECs. In this section, we illustrate its operational significance in two paradigmatic scenarios: first, the standard Rashba–Dresselhaus texture, where our framework recovers the phenomenology of Chern-number-induced obstructions; second, a configuration with vanishing net Chern flux, where the mixed-rank invariant $r^\sharp$ reveals topological features inaccessible to traditional global invariants.

7.1 Example 1: Rashba–Dresselhaus texture and the maximal rank $r = 1$

Consider a two-dimensional, two-level Hamiltonian of the Rashba–Dresselhaus (RD) type:

$$\hat{H}_{\rm RD}(k) \;=\; \frac{1}{2m}\big(\mathbf{p} - \mathbf{A}_{\rm syn}(k)\big)^2 \;+\; \mathbf{B}_{\rm syn}(k)\cdot\boldsymbol{\sigma}, \qquad k = (k_x, k_y) \in T^2_{\rm BZ},$$

where $\mathbf{A}_{\rm syn}(k)$ and $\mathbf{B}_{\rm syn}(k)$ represent the synthetic vector potential and Zeeman field, respectively. For generic coupling parameters κ and Ω, the lower dressed band induces a spin texture map $\mathbf{n} : T^2_{\rm BZ} \to S^2$.

The associated eigenline bundle $\mathbb{L} \to T^2_{\rm BZ}$ has a first Chern number $c_1(\mathbb{L})$ equal to the degree of $\mathbf{n}$. For a minimal RD texture with degree 1, the Berry curvature F represents a non-trivial cohomology class $[F] \in H^2(T^2_{\rm BZ}; \mathbb{Z})$.

In the extended parameter space $M = T^2_{\rm BZ} \times S^1_{\phi_+} \times S^1_{\phi_-}$, the synthetic torsion class is

$$[\omega] = [F] \otimes [d\phi_+] \;+\; [F] \otimes [d\phi_-] \in H^3(M; \mathbb{R}).$$

Since $\dim H^2(T^2_{\rm BZ}) = 1$, the two summands are linearly dependent, and the class factors as a single simple tensor:

$$[\omega] = [F] \otimes \big([d\phi_+] + [d\phi_-]\big).$$

Hence, the mixed tensor rank is $r = 1$. For the product metric g_0, the obstruction kernel has $\dim \mathcal{K}_0 = 1$, giving $r^\sharp_0 = 0$. However, for the physical Kaluza–Klein metric g_M (or any g_ε with $\varepsilon > 0$), Theorem3.14 shows $\mathcal{K} = 0$, so $r^\sharp = 1$. By the direct curvature analysis (Section4) we obtain

$$\dim \mathfrak{hol}^{\rm off}(\nabla^C) \;\geq\; 1.$$

Remark 7.1 (Local vs. global structure)**.** *When $c_1(\mathbb{L}) \neq 0$, the underlying principal bundle is topologically non-trivial. Nevertheless, our geometric analysis is performed in local trivializations, where M is diffeomorphic to T^4 and the metric takes the form* (2.1)*. The deformation argument and the pointwise injectivity of the obstruction map are local in nature and can be applied in each trivialization. Thus, the bound* $\dim \mathfrak{hol}^{\rm off} \geq 1$ *holds locally, detecting the irreducible coupling between momentum and phase directions even when the global bundle is non-trivial.*

7.2 Example 2: Topological persistence under vanishing net Chern flux

The power of the mixed-rank invariant becomes most apparent when global topological charges cancel. Consider a configuration where the two $U(1)$ sectors carry Berry curvatures with equal and opposite integrated fluxes:

$$\frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F^{(+)} = +1, \qquad \frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F^{(-)} = -1. \tag{7.1}$$

Then the total Chern number $c_1^{\mathrm{tot}} = c_1^{(+)} + c_1^{(-)} = 0$. Conventional Chern-number diagnostics would classify this regime as topologically trivial, yet our bound reveals a residual topological obstruction.

Under Assumption2.3 we have $F^{(\pm)} = c^{(\pm)}\,\mathrm{vol}_{\mathrm{BZ}}$ with $c^{(+)} = -c^{(-)}$. The cohomology classes satisfy $[F^{(-)}] = -[F^{(+)}]$. Hence, the torsion class becomes

$$[\omega] = [F^{(+)}]\otimes[d\phi_+] + [F^{(-)}]\otimes[d\phi_-] = [F^{(+)}]\otimes\big([d\phi_+] - [d\phi_-]\big),$$

which again has mixed rank $r = 1$. Since $F^{(\pm)} \neq 0$, the obstruction kernel vanishes for the physical metric, giving $r^\sharp = 1$. Theorem5.1 therefore guarantees

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \geq 1,$$

even though $c_1^{\mathrm{tot}} = 0$.

To see the bound concretely, choose the explicit connections

$$A^{(+)} = \frac{1}{4\pi}(k_x dk_y - k_y dk_x), \qquad A^{(-)} = -A^{(+)},$$

which satisfy (7.1) on a torus of period 2π in each direction. Write $c := c^{(+)} = 1/(2\pi)$, so that $c^{(-)} = -c$. The torsion 3-form in adapted gauge is

$$T = c\, dk_x \wedge dk_y \wedge (d\phi_+ - d\phi_-).$$

Following the method of AppendixB, we evaluate the curvature on *mixed inputs* (one horizontal, one vertical). The torsion endomorphisms give, at leading order, the quadratic torsion brackets:

$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big):\ e_1 \mapsto \tfrac{c^2}{4}(e_3 - e_4), \quad e_3 \mapsto -\tfrac{c^2}{4}\,e_1, \quad e_4 \mapsto \tfrac{c^2}{4}\,e_1,$$
$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_2}, T_{e_3}]\big):\ e_2 \mapsto \tfrac{c^2}{4}(e_3 - e_4), \quad e_3 \mapsto -\tfrac{c^2}{4}\,e_2, \quad e_4 \mapsto \tfrac{c^2}{4}\,e_2.$$

These two off-diagonal operators are *linearly independent* in $\mathrm{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \mathrm{Hom}(\mathcal{V}_p, \mathcal{H}_p)$: the first involves e_1 while the second involves e_2. With $c = 1/(2\pi)$, the coefficient is $c^2/4 = 1/(16\pi^2) \neq 0$, confirming that in this specific geometry

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C)\ \geq\ 2\ >\ 1\ =\ r^\sharp.$$

Remark 7.2 (Sharpness of the bound)**.** *The explicit computation confirms that the bound* $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ *is sharp in the sense that it cannot be improved without additional assumptions: there exist geometries (like the product metric* g_0*) where the off-diagonal holonomy is trivial, while for generic Kaluza–Klein metrics the bound is non-trivial and often exceeded. In the example above, two linearly independent mixed-input curvature operators generate at least a 2-dimensional off-diagonal holonomy algebra, illustrating that* $r^\sharp = 1$ *is a lower bound; particular geometries may exhibit a richer off-diagonal holonomy.*

Chern number c_1^{tot}	*Mixed rank* $r^\sharp$
Global aggregate of curvature	Local algebraic measure of coupling
Quantifies net vorticity	Quantifies irreducible entanglement
Can vanish by cancellation	Non-zero if each factor curvature is non-zero
Detects topological charge	Detects topological locking

Table 2: Comparison of topological invariants.

7.3 Comparison: Global invariants vs. mixed rank

The distinction between c_1^{tot} and $r^\sharp$ reflects a fundamental dichotomy between integrated and structural topology:

The mixed rank r counts the minimal number of simple tensors needed to represent the cohomology class $[\omega]$ in the Künneth decomposition. In our geometry, $r = 1$ whenever both individual Berry curvature classes $[F^{(\pm)}]$ are non-zero (and hence proportional, because $\dim H^2(T^2) = 1$). Thus r measures the irreducible geometric coupling between the Brillouin-zone torus and the phase fibres. The bound $\dim \mathfrak{hol}^{\mathrm{off}} \geq r^\sharp$ therefore constrains the algebraic complexity of the curvature, ensuring that the synthetic gauge field remains "locked" in a non-trivial configuration even when the net topological charge vanishes.

7.4 Summary

The Rashba–Dresselhaus example shows that the mixed-rank framework is consistent with established Chern-number results, while the $c_1^{\mathrm{tot}} = 0$ example demonstrates that it provides a finer classification, detecting locally non-removable curvature that conventional global invariants miss. This makes $r^\sharp$ a superior tool for analyzing the stability of synthetic gauge fields in complex SOC BEC architectures, especially in compensated regimes where global topological charges cancel but local geometric obstructions persist.

8 Beyond the Chern Number: Local Geometric Constraints and Prospective Experimental Protocols

While the Chern number c_1 quantifies the net vorticity through a global integral over the Brillouin zone, the lower bound on the off-diagonal holonomy dimension, $r^\sharp$, acts as a local structural constraint. It obstructs the simultaneous annihilation of Berry curvature components along independent directions within the product geometry. This phenomenon can be visualized as the Berry curvature being constrained to move along topological "rail-tracks" in the configuration space: even when the total topological charge is tuned to zero, specific directional components remain geometrically locked and cannot be erased by any smooth gauge transformation.

In the two-component SOC BEC manifold $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, the bound $r^\sharp = 1$ is non-degenerate. Under Assumption2.3, the constant, non-zero Berry curvatures break the integrability of the horizontal distribution, ensuring $\mathcal{K} = 0$. Consequently, the bound holds directly for the physical Kaluza–Klein metric g_M. The curvature of any metric connection calibrated by the mixed class $[\omega]$ is thus irreducible to a block-diagonal form; at least one independent off-diagonal operator persists, fundamentally mixing momentum dynamics with the phase evolution of ϕ_+ and ϕ_-.

This geometric prediction suggests concrete interferometric tests. Consider the simultaneous measurement of the Berry phases $\Phi_B^+(\gamma)$ and $\Phi_B^-(\gamma)$ accumulated along a closed loop $\gamma \subset T^2_{\mathrm{BZ}}$ by separately addressing the global and relative phase sectors. If the gauge field were fully reducible (i.e., if the holonomy were block-diagonal with respect to the product splitting), one could choose a gauge where both phases vanish for every γ. The bound $r^\sharp = 1$ forbids such a simultaneous vanishing across all loops. For any pair of independent cycles $\{\gamma_1, \gamma_2\}$ in T^2_{BZ}, the two-component Berry-phase vector:

$$\mathbf{\Phi}_B = \big(\Phi_B^+(\gamma_1), \Phi_B^-(\gamma_1), \Phi_B^+(\gamma_2), \Phi_B^-(\gamma_2)\big)$$

cannot be continuously deformed to zero while preserving the mixed cohomology class $[\omega]$.

Such measurements are within the reach of current cold-atom experimental capabilities. Interferometric techniques used to map Berry curvature in Hofstadter bands[2] or to probe spin-texture topology in SOC systems[9] can be adapted to track the two phase degrees of freedom $\phi_\pm$ independently[5]. The required control can be implemented via optical "painting"[8], allowing for precise spatiotemporal manipulation of the condensate's phase profile.

Observing that these Berry-phase components cannot be simultaneously flattened would provide the first direct experimental signature of the topological obstruction captured by $r^\sharp$. This demonstrates that the mixed-cohomology framework not only yields a rigorous mathematical lower bound but also provides a feasible protocol for detecting locally irremovable Berry curvature, effectively moving the study of quantum gases beyond the global Chern-number paradigm.

9 Conclusions

We have demonstrated that a two-component spin–orbit-coupled (SOC) BEC possesses an intrinsic geometric obstruction that prevents the simultaneous flattening of Berry curvature along the independent phase directions $\phi_\pm$. This obstruction persists even when the net global Chern flux vanishes, identifying a topological regime that eludes traditional integrated invariants.

The extended parameter space $M \cong T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, equipped with the physical Kaluza–Klein metric g_M induced by synthetic gauge fields, carries a metric connection ∇^C with totally skew-symmetric torsion. Under Assumption2.3 (constant Berry curvatures), we have established that the mixed tensor rank of the torsion class is $r = 1$.

The core of our proof lies in the resolution of the Kaluza–Klein kernel problem. By introducing a smooth deformation family of metrics g_ε (Definition2.5), we have shown that while the original PT bound is trivial for the product metric ($\varepsilon = 0$), the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes for any $\varepsilon > 0$ due to the non-parallelism of the vertical coframe (Theorem2.14). By applying Theorem 4.7, we transferred the non-trivial bound from the deformed cases to the physical endpoint $\varepsilon = 1$. This yields the sharp inequality:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \;\geq\; 1. \tag{9.1}$$

This algebraic constraint implies that momentum degrees of freedom are fundamentally "locked" to the phase sectors, an irreducible coupling that cannot be removed by smooth gauge transformations. The persistence of the bound in regimes with zero total Chern number—as illustrated in Section7—highlights that the mixed-rank invariant $r^\sharp$ operates beyond the Chern-number paradigm, detecting structural features of the gauge field that integrated invariants fail to resolve.

Experimentally, this framework suggests interferometric protocols to resolve independent Berry phases (Section8), providing a direct signature of locally irremovable curvature. Conceptually, it establishes a new mechanism for geometric robustness in synthetic gauge systems, controlled by the algebraic structure of mixed cohomology components.

In conclusion, beyond the specific application to spin–orbit-coupled BECs, the deformation strategy developed here—reducing a Kaluza–Klein geometry to its product limit while tracking the evolution of the obstruction kernel—provides a general template for extending the Pigazzini-Toda lower bound [12] to fibered geometries beyond the Riemannian product setting. The essential requirement is that the parallel-form strata be computable for the physical metric, enabling explicit evaluation of the reduced rank $r^\sharp$. We expect this approach to be applicable to other physical systems where synthetic gauge fields induce non-trivial geometric phases, including photonic topological insulators, superconducting qubit arrays, and higher-dimensional Kaluza–Klein compactifications in string theory.

Acknowledgements

The authors gratefully acknowledge stimulating discussions with colleagues in geometry and cold-atom physics that helped shape the formulation of these results.

A Levi-Civita Connection of the Deformed Metric

In this appendix, we derive the connection 1-forms and the covariant derivatives for the deformation family g_ε defined in (2.1). We work in a local coordinate system (x, y, ϕ_+, ϕ_-) on M such that $g_{\mathrm{BZ}} = dx^2 + dy^2$ and $F^{(\pm)} = c^{(\pm)} dx \wedge dy$.

A.1 Orthonormal Coframe

For a fixed $\varepsilon \in [0, 1]$, an orthonormal coframe $\{e^a\}$ for g_ε is:

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = d\phi_+ + \varepsilon A^{(+)}, \quad e^4 = d\phi_- + \varepsilon A^{(-)}.$$

The structure equations $de^a + \omega^a_{\ b} \wedge e^b = 0$ (with $\omega_{ab} = -\omega_{ba}$) determine the connection 1-forms.

Taking the exterior derivative of e^3 and e^4:

$$de^3 = \varepsilon dA^{(+)} = \varepsilon c^{(+)} e^1 \wedge e^2, \quad de^4 = \varepsilon dA^{(-)} = \varepsilon c^{(-)} e^1 \wedge e^2.$$

The non-zero connection 1-forms are:

$$\omega_{31} = -\omega_{13} = -\frac{\varepsilon c^{(+)}}{2} e^2, \quad \omega_{32} = -\omega_{23} = \frac{\varepsilon c^{(+)}}{2} e^1,$$
$$\omega_{41} = -\omega_{14} = -\frac{\varepsilon c^{(-)}}{2} e^2, \quad \omega_{42} = -\omega_{24} = \frac{\varepsilon c^{(-)}}{2} e^1.$$

A.2 Covariant Derivatives of the Coframe

The covariant derivative of the vertical 1-forms is $\nabla^{LC_\varepsilon} e^a = -\omega^a_{\ b} \otimes e^b$.

For e^3:

$$\nabla^{LC_\varepsilon} e^3 = \frac{\varepsilon c^{(+)}}{2}(e^2 \otimes e^1 - e^1 \otimes e^2). \tag{A.1}$$

Similarly for e^4:

$$\nabla^{LC_\varepsilon} e^4 = \frac{\varepsilon c^{(-)}}{2}(e^2 \otimes e^1 - e^1 \otimes e^2). \tag{A.2}$$

Note that for $\varepsilon = 0$ (product metric), $\nabla^{LC_0} e^3 = \nabla^{LC_0} e^4 = 0$, confirming that vertical forms are parallel in the product case. For $\varepsilon > 0$, the vertical coframe is no longer parallel.

A.3 Christoffel Symbols

A.3.1 Computation of $\Gamma^{\phi_+}_{xy}$

We now derive the explicit formula for the mixed Christoffel symbol using the connection 1-forms computed above.

Direct computation from connection 1-forms. At point p in adapted gauge, the Christoffel symbols are related to the connection 1-forms by:

$$\Gamma^{\phi_+}_{xy}\big|_p = \Gamma^3_{12}\big|_p = \omega_{32}(\partial_x)\big|_p, \tag{A.3}$$

where we use the coordinate identification $x^1 = x, x^2 = y, x^3 = \phi_+$.

From the connection 1-forms derived above:

$$\omega_{32} = \frac{\varepsilon c^{(+)}}{2} e^1 = \frac{\varepsilon c^{(+)}}{2} dx. \tag{A.4}$$

Evaluating on ∂_x:

$$\omega_{32}(\partial_x)\big|_p = \frac{\varepsilon c^{(+)}}{2}\, dx(\partial_x) = \frac{\varepsilon c^{(+)}}{2}. \tag{A.5}$$

Therefore:

$$\Gamma^{\phi_+}_{xy}\big|_p = \frac{\varepsilon c^{(+)}}{2}. \tag{A.6}$$

Antisymmetric part and torsion verification. For the other mixed component:

$$\Gamma^{\phi_+}_{yx}\big|_p = \omega_{32}(\partial_y)\big|_p = \frac{\varepsilon c^{(+)}}{2}\, dx(\partial_y) = 0. \tag{A.7}$$

The antisymmetric part (related to torsion) is:

$$\Gamma^{\phi_+}_{xy} - \Gamma^{\phi_+}_{yx} = \frac{\varepsilon c^{(+)}}{2} - 0 = \frac{\varepsilon c^{(+)}}{2}. \tag{A.8}$$

Note that the full torsion 3-form T_ε evaluated gives:

$$T_\varepsilon(\partial_x, \partial_y, \partial_{\phi_+})\big|_p = c^{(+)}, \tag{A.9}$$

which differs from the Christoffel antisymmetric part by a factor. This is because the connection ∇^C with torsion (Proposition2.12) has the relationship:

$$\nabla^C_X Y = \nabla^{LC}_X Y + \frac{1}{2} T(X, Y, \cdot)^\sharp, \tag{A.10}$$

where the factor 1/2 and the metric raising of the index account for the difference. The Christoffel symbols computed here are those of the Levi-Civita connection ∇^{LC}, which enter the curvature formula used in Theorem4.5.

Final result. The mixed Christoffel symbols are *exact* (no higher-order corrections):

$$\Gamma^{\phi_+}_{xy}\big|_p = \frac{\varepsilon c^{(+)}}{2}, \tag{A.11}$$

$$\Gamma^{\phi_-}_{xy}\big|_p = \frac{\varepsilon c^{(-)}}{2}, \tag{A.12}$$

with all other mixed symbols vanishing. These are exact because the connection 1-forms ω_{3a}, ω_{4a} are linear in ε with constant coefficients (under Assumption2.3), and no higher-order terms arise in the structure equations.

These formulas provide the explicit Christoffel symbols required for the curvature computation in Theorem4.5 and AppendixB.

B Curvature Computation for the SOC BEC Model

We establish the non-vanishing of *off-diagonal* curvature for the connection ∇^{C_ε}, thereby justifying Theorem4.5. The computation uses the adapted gauge of AppendixA and exploits the homogeneity of the model under Assumption2.3.

B.1 Notation and curvature decomposition

At a point $p \in M$ in adapted gauge, the orthonormal frame is

$$e_1 = \partial_x, \quad e_2 = \partial_y \qquad \text{(horizontal, } \mathcal{H}_p), \qquad e_3 = \partial_{\phi_+}, \quad e_4 = \partial_{\phi_-} \qquad \text{(vertical, } \mathcal{V}_p).$$

Definition B.1 (Off-diagonal endomorphism)**.** *An endomorphism $A \in \mathfrak{so}(T_pM)$ is called* off-diagonal *(with respect to the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$) if $A(\mathcal{H}_p) \subset \mathcal{V}_p$ and $A(\mathcal{V}_p) \subset \mathcal{H}_p$. We write $\pi_{\text{off}}(A)$ for the off-diagonal projection ofA.*

The curvature of $\nabla^{C_\varepsilon} = \nabla^{\text{LC}(\varepsilon)} + \frac{1}{2} T_\varepsilon$ decomposes as

$$R^{C_\varepsilon}(X, Y) \;=\; R^{\text{LC}(\varepsilon)}(X, Y) \;+\; \tfrac{1}{2}\big[(\nabla^{\text{LC}(\varepsilon)}_X T_\varepsilon)(Y, \cdot) - (\nabla^{\text{LC}(\varepsilon)}_Y T_\varepsilon)(X, \cdot)\big]^\sharp \;+\; \tfrac{1}{4}[T_X, T_Y], \tag{B.1}$$

where $T_X(Z) := T_\varepsilon(X, Z, \cdot)^\sharp$ and $[T_X, T_Y] := T_X \circ T_Y - T_Y \circ T_X$.

To produce an off-diagonal holonomy generator we evaluate the curvature on *mixed inputs*: one horizontal, one vertical. Concretely, we compute

$$\pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big) \;\in\; \text{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \text{Hom}(\mathcal{V}_p, \mathcal{H}_p).$$

B.2 Homogeneity of the model

Proposition B.2 (Homogeneity)**.** *Under Assumption2.3, the curvature tensor R^{C_ε} is translation-invariant on $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$. In particular, its components in the orthonormal frame $\{e_1, e_2, e_3, e_4\}$ are independent of the point $p \in M$.*

Proof. The torsion 3-form at p in adapted gauge reads $T_\varepsilon|_p = c^{(+)} e^1 \wedge e^2 \wedge e^3 + c^{(-)} e^1 \wedge e^2 \wedge e^4$, with *constant* coefficients $c^{(\pm)}$. The connection 1-forms (AppendixA) are $\omega_{31} = -\frac{\varepsilon c^{(+)}}{2} e^2$, $\omega_{32} = \frac{\varepsilon c^{(+)}}{2} e^1$, $\omega_{41} = -\frac{\varepsilon c^{(-)}}{2} e^2$, $\omega_{42} = \frac{\varepsilon c^{(-)}}{2} e^1$, all with constant coefficients. Since the curvature 2-forms $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ and the torsion endomorphisms T_X are computed from these constant data, every component of R^{C_ε} is independent of p. □

Corollary B.3 (Pointwise implies global)**.** *If $\pi_{\mathrm{off}}(R^{C_\varepsilon}(e_i, e_\alpha)) \neq 0$ at one point $p_0 \in M$, then the same holds at every $p \in M$.*

B.3 Torsion endomorphisms

The torsion at p in adapted gauge is

$$T_\varepsilon|_p \;=\; c^{(+)}\, e^1 \wedge e^2 \wedge e^3 \;+\; c^{(-)}\, e^1 \wedge e^2 \wedge e^4.$$

This expression is *independent of* ε at$p$ (because $A^{(\pm)}|_p = 0$ in adapted gauge).

Lemma B.4 (Torsion endomorphism table)**.** *The nonzero values of the endomorphism $T_{e_a}(e_b) = T(e_a, e_b, \cdot)^\sharp$ are:*

$T_{e_a}(e_b)$	$b=1$	$b=2$	$b=3$	$b=4$
$a=1$	0	$c^{(+)}e_3 + c^{(-)}e_4$	$-c^{(+)}e_2$	$-c^{(-)}e_2$
$a=2$	$-c^{(+)}e_3 - c^{(-)}e_4$	0	$c^{(+)}e_1$	$c^{(-)}e_1$
$a=3$	$c^{(+)}e_2$	$-c^{(+)}e_1$	0	0
$a=4$	$c^{(-)}e_2$	$-c^{(-)}e_1$	0	0

Proof. Direct computation from $T(e_a, e_b, e_c) = c^{(+)}(e^1 \wedge e^2 \wedge e^3)(e_a, e_b, e_c) + c^{(-)}(e^1 \wedge e^2 \wedge e^4)(e_a, e_b, e_c)$ and the definition $T_{e_a}(e_b) = \sum_c T(e_a, e_b, e_c)\, e_c$. □

Remark B.5 (Type analysis)**.** *Observe from the table:*

- $T_{e_1}(e_2) = c^{(+)}e_3 + c^{(-)}e_4 \in \mathcal{V}$ *($\mathcal{H} \to \mathcal{V}$: off-diagonal).*
- $T_{e_1}(e_3) = -c^{(+)}e_2 \in \mathcal{H}$ *($\mathcal{V} \to \mathcal{H}$: off-diagonal).*
- $T_{e_3}(e_1) = c^{(+)}e_2 \in \mathcal{H}$ *($\mathcal{H} \to \mathcal{H}$: diagonal).*

Thus the torsion itself *already mixes horizontal and vertical directions. The quadratic torsion term $\frac{1}{4}[T_X, T_Y]$ inherits this mixing.*

B.4 Quadratic torsion for mixed inputs

We compute the commutator $[T_{e_1}, T_{e_3}]$ using the table of LemmaB.4.

Proposition B.6. *The commutator $[T_{e_1}, T_{e_3}] \in \mathfrak{gl}(T_pM)$ is given by:*

$$[T_{e_1}, T_{e_3}](e_1) = (c^{(+)})^2 \, e_3 + c^{(+)}c^{(-)} \, e_4, \tag{B.2}$$

$$[T_{e_1}, T_{e_3}](e_2) = 0, \tag{B.3}$$

$$[T_{e_1}, T_{e_3}](e_3) = -(c^{(+)})^2 \, e_1, \tag{B.4}$$

$$[T_{e_1}, T_{e_3}](e_4) = -c^{(+)}c^{(-)} \, e_1. \tag{B.5}$$

Proof. We compute each entry using $[T_{e_1}, T_{e_3}](e_c) = T_{e_1}(T_{e_3}(e_c)) - T_{e_3}(T_{e_1}(e_c))$:

Action on e_1: $T_{e_3}(e_1) = c^{(+)}e_2$. Then $T_{e_1}(c^{(+)}e_2) = c^{(+)}(c^{(+)}e_3 + c^{(-)}e_4)$. Also $T_{e_1}(e_1) = 0$, so $T_{e_3}(T_{e_1}(e_1)) = 0$. Result: $(c^{(+)})^2 e_3 + c^{(+)}c^{(-)}e_4$.

Action on e_2: $T_{e_3}(e_2) = -c^{(+)}e_1$. Then $T_{e_1}(-c^{(+)}e_1) = 0$. Also $T_{e_1}(e_2) = c^{(+)}e_3 + c^{(-)}e_4$, $T_{e_3}(c^{(+)}e_3 + c^{(-)}e_4) = 0$. Result: 0.

Action on e_3: $T_{e_3}(e_3) = 0$, so the first term vanishes. $T_{e_1}(e_3) = -c^{(+)}e_2$, $T_{e_3}(-c^{(+)}e_2) = -c^{(+)}(-c^{(+)}e_1) = (c^{(+)})^2 e_1$. Result: $0 - (c^{(+)})^2 e_1 = -(c^{(+)})^2 e_1$.

Action on e_4: $T_{e_3}(e_4) = 0$, so the first term vanishes. $T_{e_1}(e_4) = -c^{(-)}e_2$, $T_{e_3}(-c^{(-)}e_2) = -c^{(-)}(-c^{(+)}e_1) = c^{(+)}c^{(-)}e_1$. Result: $0 - c^{(+)}c^{(-)}e_1 = -c^{(+)}c^{(-)}e_1$. □

Corollary B.7 (Off-diagonal projection)**.** *The off-diagonal part of $[T_{e_1}, T_{e_3}]$ is:*

$$\pi_{\text{off}}\big([T_{e_1}, T_{e_3}]\big) \;:\; \begin{cases} e_1 \;\mapsto\; (c^{(+)})^2 \, e_3 + c^{(+)}c^{(-)} \, e_4 & (\mathcal{H} \to \mathcal{V}), \\ e_2 \;\mapsto\; 0, \\ e_3 \;\mapsto\; -(c^{(+)})^2 \, e_1 & (\mathcal{V} \to \mathcal{H}), \\ e_4 \;\mapsto\; -c^{(+)}c^{(-)} \, e_1 & (\mathcal{V} \to \mathcal{H}). \end{cases} \tag{B.6}$$

Every entry in (B.6) *is already off-diagonal (no projection is lost), and the map is nonzero whenever $c^{(+)} \neq 0$.*

B.5 Contribution of the remaining curvature terms

The curvature $R^{C_\varepsilon}(e_1, e_3)$ also receives contributions from $R^{\text{LC}(\varepsilon)}$ and from the torsion derivative $\frac{1}{2}[(\nabla_{e_1}T)(e_3, \cdot) - (\nabla_{e_3}T)(e_1, \cdot)]^\sharp$. We now verify that these are of lower order.

Lemma B.8 (Levi-Civita curvature for mixed inputs)**.** *For mixed inputs (e_1, e_3), the Levi-Civita curvature vanishes to leading order:*

$$R^{\text{LC}(\varepsilon)}(e_1, e_3) = 0. \tag{B.7}$$

Proof. The curvature 2-forms computed in AppendixA are all proportional to $e^1 \wedge e^2$:

$$\Omega^1{}_2 = -\tfrac{\varepsilon^2}{4}\big[(c^{(+)})^2 + (c^{(-)})^2\big] \, e^1 \wedge e^2,$$

and $\Omega^3{}_1 = \Omega^3{}_2 = \Omega^4{}_1 = \Omega^4{}_2 = \Omega^3{}_4 = 0$. Evaluating any $\Omega^a{}_b$ on (e_1, e_3) gives zero, since $e^1 \wedge e^2(e_1, e_3) = 0$. □

Lemma B.9 (Torsion derivative for mixed inputs)**.** *The torsion-derivative contribution to $R^{C_\varepsilon}(e_1, e_3)$ is of order $\mathcal{O}(\varepsilon)$:*

$$\tfrac{1}{2}\big[(\nabla_{e_1} T_\varepsilon)(e_3, \cdot) - (\nabla_{e_3} T_\varepsilon)(e_1, \cdot)\big]^\sharp = \mathcal{O}(\varepsilon). \tag{B.8}$$

Proof. The torsion $T_\varepsilon|_p = c^{(+)} e^1 \wedge e^2 \wedge e^3 + c^{(-)} e^1 \wedge e^2 \wedge e^4$ has constant coefficients in adapted gauge. Its covariant derivative $(\nabla_X T)(Y, Z, W)$ involves Christoffel corrections applied to each argument. Since the only nonzero connection coefficients are of order ε (specifically, $\omega_{3a} \sim \varepsilon c^{(+)}$ and $\omega_{4a} \sim \varepsilon c^{(-)}$), the covariant derivative ∇T is $\mathcal{O}(\varepsilon)$. □

B.6 Main result: non-vanishing off-diagonal curvature

Theorem B.10 (Off-diagonal curvature for mixed inputs)**.** *For all $c^{(\pm)} \neq 0$ and $\varepsilon > 0$, the off-diagonal curvature at every point $p \in M$ is nonzero:*

$$\pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big) \;\neq\; 0 \qquad \textit{for all } p \in M. \tag{B.9}$$

Proof. We evaluate the three terms in the curvature decomposition(B.1) for the mixed inputs $X = e_1 \in \mathcal{H}_p$, $Y = e_3 \in \mathcal{V}_p$.

Term1 (Levi-Civita): By LemmaB.8, $R^{\text{LC}(\varepsilon)}(e_1, e_3) = 0$.

Term2 (Torsion derivative): By LemmaB.9, this is $\mathcal{O}(\varepsilon)$.

Term3 (Quadratic torsion): By PropositionB.6, the off-diagonal projection(B.6) gives at $\mathcal{O}(1)$:

$$\pi_{\text{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big) \;:\quad e_1 \mapsto \tfrac{(c^{(+)})^2}{4}\, e_3 + \tfrac{c^{(+)}c^{(-)}}{4}\, e_4, \qquad e_3 \mapsto -\tfrac{(c^{(+)})^2}{4}\, e_1. \tag{B.10}$$

For $c^{(+)} \neq 0$, the map $e_3 \mapsto -\frac{(c^{(+)})^2}{4} e_1$ is nonzero.

Combining: The off-diagonal part of the curvature is

$$\pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big) \;=\; \pi_{\text{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big) \;+\; \mathcal{O}(\varepsilon).$$

We now show that the result is *exact*, not merely asymptotic. The Levi-Civita curvature vanishes on the mixed pair: $R^{LC(\varepsilon)}(e_1, e_3) = 0$ (Lemma B.8). For the torsion-derivative term, observe that all connection 1-forms $\omega^a{}_b$ are proportional to e^1 or e^2 (Appendix A.1), hence $\omega^a{}_b(e_3) = 0$ for every a, b. It follows that $\nabla^{LC(\varepsilon)}_{e_3} = 0$ on constant-coefficient tensors, giving $(\nabla^{LC(\varepsilon)}_{e_3} T_\varepsilon)(e_1, \cdot) = 0$. For the other summand, antisymmetry of T yields $T_\varepsilon(e_3, e_3, \cdot) = 0$, so $(\nabla^{LC(\varepsilon)}_{e_1} T_\varepsilon)(e_3, e_3, \cdot)$ receives no contribution from the slot containing e_3. A direct computation using the Christoffel corrections on the remaining slots shows that the off-diagonal projection of the torsion-derivative term, evaluated on the triple $(e_1, e_3; e_3)$, vanishes identically.

Combining the three contributions, we obtain the *exact* identity, valid for all $\varepsilon \in [0, 1]$:

$$\big\langle e_1,\, \pi_{\text{off}}\big(R^{C_\varepsilon}(e_1, e_3)\big)\, e_3 \big\rangle \;=\; -\frac{(c^{(+)})^2}{4} \;\neq\; 0 \qquad \text{for all } c^{(+)} \neq 0. \tag{B.11}$$

By homogeneity (PropositionB.2), the result holds at every point $p \in M$. □

Remark B.11 (Role of each curvature input type). *The computation reveals a crucial structural point: the off-diagonal holonomy generator arises from the curvature with* mixed *inputs* $R^C(e_1, e_3)$ *(one horizontal, one vertical), not from the horizontal-input curvature* $R^C(e_1, e_2)$*.*

The curvature $R^C(e_1, e_2)$ *restricted to* $\mathcal{V}_p$ *measures the* $\mathfrak{so}(\mathcal{V})$*-component of the holonomy (the* diagonal *part, which rotates fiber directions among themselves). This component is also nonzero but does not contribute to* $\mathfrak{hol}^{\mathrm{off}}$*.*

In contrast, $R^C(e_1, e_3)$ *evaluated on the basis vectors produces endomorphisms that map* $\mathcal{H} \to \mathcal{V}$ *and* $\mathcal{V} \to \mathcal{H}$*, which is precisely the* $\mathrm{Hom}(\mathcal{H}, \mathcal{V}) \oplus \mathrm{Hom}(\mathcal{V}, \mathcal{H})$ *component of* $\mathfrak{so}(T_pM)$ *— the off-diagonal holonomy algebra* $\mathfrak{hol}^{\mathrm{off}}$ *in the sense of[12].*

B.7 Global off-diagonal holonomy bound

Corollary B.12 (Global off-diagonal holonomy bound). *For all* $c^{(\pm)} \neq 0$ *and* $\varepsilon \in (0, 1]$*:*

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \ \geq \ 1 \qquad \textit{for every } p \in M. \tag{B.12}$$

Proof. By TheoremB.10, the off-diagonal curvature operator $\pi_{\mathrm{off}}(R^{C_\varepsilon}(e_1, e_3)) \neq 0$ at every point $p \in M$. By the Ambrose–Singer theorem, this operator belongs to $\mathfrak{hol}_p(\nabla^{C_\varepsilon})$. Its off-diagonal projection is nonzero by construction, so $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \neq \{0\}$, yielding $\dim \mathfrak{hol}_p^{\mathrm{off}} \geq 1$. □

B.8 Explicit off-diagonal curvature for all mixed inputs

For completeness, we record the leading-order off-diagonal projection for all four mixed-input curvatures.

Proposition B.13 (All mixed-input off-diagonal curvatures). *At leading order* $\mathcal{O}(1)$*, the off-diagonal projections of the quadratic torsion term for all mixed input pairs are:*

$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_3}]\big) : \ e_1 \mapsto \tfrac{(c^{(+)})^2}{4} e_3 + \tfrac{c^{(+)}c^{(-)}}{4} e_4, \quad e_3 \mapsto -\tfrac{(c^{(+)})^2}{4} e_1, \quad e_4 \mapsto -\tfrac{c^{(+)}c^{(-)}}{4} e_1, \tag{B.13}$$

$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_1}, T_{e_4}]\big) : \ e_1 \mapsto \tfrac{c^{(+)}c^{(-)}}{4} e_3 + \tfrac{(c^{(-)})^2}{4} e_4, \quad e_3 \mapsto -\tfrac{c^{(+)}c^{(-)}}{4} e_1, \quad e_4 \mapsto -\tfrac{(c^{(-)})^2}{4} e_1, \tag{B.14}$$

$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_2}, T_{e_3}]\big) : \ e_2 \mapsto \tfrac{(c^{(+)})^2}{4} e_3 + \tfrac{c^{(+)}c^{(-)}}{4} e_4, \quad e_3 \mapsto -\tfrac{(c^{(+)})^2}{4} e_2, \quad e_4 \mapsto -\tfrac{c^{(+)}c^{(-)}}{4} e_2, \tag{B.15}$$

$$\pi_{\mathrm{off}}\big(\tfrac{1}{4}[T_{e_2}, T_{e_4}]\big) : \ e_2 \mapsto \tfrac{c^{(+)}c^{(-)}}{4} e_3 + \tfrac{(c^{(-)})^2}{4} e_4, \quad e_3 \mapsto -\tfrac{c^{(+)}c^{(-)}}{4} e_2, \quad e_4 \mapsto -\tfrac{(c^{(-)})^2}{4} e_2. \tag{B.16}$$

In particular:

- *If* $c^{(+)} \neq 0$*: the pairs* (e_1, e_3) *and* (e_2, e_3) *yield nonzero off-diagonal operators.*
- *If* $c^{(-)} \neq 0$*: the pairs* (e_1, e_4) *and* (e_2, e_4) *yield nonzero off-diagonal operators.*
- *For* $c^{(+)} \neq 0$ *and* $c^{(-)} \neq 0$*: all four pairs produce nonzero off-diagonal curvature.*

Proof. Analogous to the proof of PropositionB.6, using the torsion table of LemmaB.4. □

B.9 Physical interpretation

Remark B.14 (Experimental significance)**.** *The homogeneity of the SOC BEC model has direct experimental consequences:*

1. *Uniform constraint: The off-diagonal curvature is nonzero at* every *point in the extended parameter space, not just on a dense open set. Any measurement of Berry-phase correlations will detect the irreducible mixing of momentum and phase degrees of freedom, regardless of the momentum-space location.*

2. *Robustness: The leading off-diagonal contribution (TheoremB.10) is of order* $\mathcal{O}(1)$ *in the deformation parameter* ε*, arising from the quadratic torsion term. This is structurally stable: it depends only on the constants* $c^{(\pm)} \neq 0$ *and cannot be removed by small perturbations.*

3. *Mixed-input origin: The off-diagonal holonomy arises from the curvature with mixed (horizontal–vertical) inputs, reflecting the physical coupling between momentum transport* ($\mathcal{H}$) *and phase evolution* ($\mathcal{V}$)*.*

References

[1] M. Aidelsburger, S. Nascimbene, N. Goldman, *Artificial gauge fields in materials and engineered systems*, C. R. Physique 19 (2018), 394–432.

[2] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, *Measuring the Chern number of Hofstadter bands with ultracold bosonic atoms*, Nature Phys. 11 (2015), 162–166.

[3] A.S.Arnold, C.S.Garvie, and E.Riis, *Large magnetic storage ring for Bose–Einstein condensates*, Phys. Rev. A *73* (2006), 041606.

[4] J. Dalibard, F.Gerbier, G.Juzeliūnas, and P.Öhberg, *Colloquium: Artificial gauge potentials for neutral atoms*, Rev. Mod. Phys. *83* (2011), 1523–1543.

[5] L. Duca, T. Li, M. Reitter, I. Bloch, M. Schleier-Smith, and E. Demler, *An Aharonov-Bohm interferometer for determining Bloch-band topology*, Science 347 (2015), 288–292.

[6] V. Galitski and I. B. Spielman, *Spin-orbit coupling in quantum gases*, Nature *494* (2013), 49–54.

[7] N. Goldman, G.Juzeliūnas, P.Öhberg, and I.B.Spielman, *Light-induced gauge fields for ultracold atoms*, Rep. Prog. Phys. *77* (2014), 126401.

[8] K. Henderson, C.Ryu, C.MacCormick, and M.G.Boshier, *Experimental demonstration of painting arbitrary and dynamic potentials for Bose–Einstein condensates*, New J. Phys. *11* (2009), 043030.

[9] Y.-J. Lin, K.Jiménez-García, and I.B.Spielman, *Spin–orbit-coupled Bose–Einstein condensates*, Nature *471* (2011), 83–86.

[10] M. W. Ray, E. Ruokokoski, S. Kandel, M. Möttönen, D. S. Hall, *Observation of Dirac monopoles in a synthetic magnetic field*, Nature 505 (2014), 657–660.

[11] M. R. Matthews, B. Benton, M. Krygier, M. Edwards, and C. W. Clark, *Realization of a double-ring BEC with tunable tunnelling*, Phys. Rev. Lett. 121 (2018).

[12] A. Pigazzini and M. Toda, *Cohomological Calibration and Curvature Constraints on Product Manifolds: A Topological Lower Bound*, arXiv:2509.11834v13.

[13] B. Simon, Holonomy, *the Quantum Adiabatic Theorem, and Berry's Phase*, Phys. Rev. Lett. 51 (1983), 2167–2170.

[14] D. M. Stamper-Kurn and M. Ueda, Spinor Bose gases: symmetries, magnetism and quantum dynamics, Rev. Mod. Phys. *85* (2013), 1191–1244.

[15] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer–Verlag, (1983).