COHOMOLOGICAL CALIBRATION AND CURVATURE CONSTRAINTS ON PRODUCT MANIFOLDS: A TOPOLOGICAL LOWER BOUND

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Abstract. We establish a quantitative relationship between mixed de Rham classes and the geometric complexity of metric connections with totally skew torsion on product manifolds where both factors are compact oriented surfaces. For any cohomologically calibrated connection ∇^C with non-parallel torsion T, where the harmonic projection of T represents a mixed class $[\omega]$, we prove that on a non-empty open subset $\mathcal{V} \subset M$,

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \geq \min\bigl(\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K},\ 1\bigr),$$

with $\mathcal{K}$ an intrinsically defined obstruction space. The bound is a topological invariant under metric deformations preserving the parallel-form strata and provides an obstruction to reducible holonomy. Counterexamples show the hypothesis is optimal.

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1. Introduction

Let $M = M_1 \times M_2$ be a compact oriented product manifold endowed with a product metric $g = g_1 \oplus g_2$, inducing at each point an orthogonal splitting $T_pM = V_1 \oplus V_2$. We introduce a topological invariant, the *mixed tensor rank* of a cohomology class $[\omega] \in H^3(M;\mathbb{R})$, and show that—after subtracting a correction term that captures the part of the class absorbed by parallel forms—it provides a lower bound for the dimension of the off-diagonal holonomy subspace generated by any *cohomologically calibrated* metric connection ∇^C with totally skew torsion.

A subtle but crucial point is that the subspaces spanned by a minimal harmonic decomposition are uniquely determined only under topological restrictions on the factors. Consequently, our main result holds for *admissible* product manifolds (Definition 2.2), i.e., where both factors are compact oriented surfaces.

For any admissible M, any cohomologically calibrated connection ∇^C with non-parallel torsion satisfies, on a non-empty open subset $\mathcal{V} \subset M$,

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \ \geq\ r^\sharp := \min\bigl(\mathrm{rank}_{\mathbb{R}}\bigl([\omega]_{\mathrm{mixed}}\bigr) - \dim \mathcal{K},\ 1\bigr),$$

where the obstruction space $\mathcal{K}$ is intrinsically defined via intersections of the mixed factor spaces $\mathcal{V}_{2,1}$, $\mathcal{V}_{1,2}$ with the parallel-form strata $\mathcal{P}_k(M_i)$. The bound is invariant under metric deformations preserving these strata.

When $r^\sharp > 0$ the holonomy representation is forced to be irreducible across the product splitting. The examples in Section 6 illustrate both non-trivial ($\Sigma_g \times T^2$) and trivial ($T^2 \times T^2$) cases, and Remark 2.4 shows by explicit counterexample that without the admissibility hypothesis the bound fails to be well-defined.

2. Cohomologically Calibrated Metric Connections with Skew Torsion

Throughout, ∇^{LC} denotes the Levi-Civita connection of g. We consider metric connections ∇^C with totally skew torsion T, i.e.

$$\nabla^C g = 0, \tag{2.1}$$

$$T(X, Y, Z) := g(T(X, Y), Z) \in \Omega^3(M), \tag{2.2}$$

and write

$$\nabla^C = \nabla^{LC} + K, \tag{2.3}$$

$$K_{XYZ} = \tfrac{1}{2} T_{XYZ}. \tag{2.4}$$

Thus $T^\flat := T$ is a 3-form. The curvature R^C of ∇^C can be expressed in terms of R^{LC}, $\nabla^{LC}T$, and quadratic torsion terms; in abstract index notation,

$$R^C_{XY}Z = R^{LC}_{XY}Z + \tfrac{1}{2}\big((\nabla^{LC}_X T)(Y, Z, \cdot)^\sharp - (\nabla^{LC}_Y T)(X, Z, \cdot)^\sharp\big) + \tfrac{1}{4}\, Q_T(X, Y)Z, \tag{2.5}$$

where Q_T is bilinear and quadratic in T (see, e.g., [4] or standard torsion-curvature formulas). We shall use (2.5) only qualitatively: mixed components of T produce mixed components of R^C through both the $\nabla^{LC}T$ and $T * T$ terms.

Definition 2.1 (Metric cohomological calibration)**.** A metric connection ∇^C with non-parallel ($\nabla^{LC}T \neq 0$) and totally skew torsion T, where the torsion 3-form represents a non trivial mixed cohomology class $[\omega]$ via its harmonic projection ($T^\flat = \omega_h + d\alpha + \delta\beta$, i.e. $[\omega] := [\omega_h]$), is called *cohomologically calibrated*. The class is *mixed* if it lies outside the natural image of $H^3(M_1) \oplus H^3(M_2)$ under the Künneth isomorphism:

$$H^3(M) \cong \bigoplus_{p+q=3} H^p(M_1) \otimes H^q(M_2). \tag{2.6}$$

Since M is compact and oriented, Hodge theory applies to g: each cohomology class has a unique harmonic representative. We denote by ω_h the harmonic representative of $[\omega]$ with respect to g.
For a thorough treatment of Hodge theory and cohomology on complex manifolds, see [6].

2.1. **Intrinsic Definition of Mixed Cohomology Subspaces.** We now address a subtle but fundamental point: the subspaces spanned by the factors in a minimal harmonic decomposition are not uniquely determined in general. To obtain a well–defined geometric invariant we must impose conditions on the topology of the factors.

Definition 2.2 (Admissible Product Manifolds)**.** A compact oriented product manifold $M = M_1 \times M_2$ is called admissible if it satisfies the following:

both factors M_1, M_2 are compact oriented surfaces.

Lemma 2.3 (Uniqueness of Mixed Subspaces)**.** *Let M be an admissible product manifold and $[\omega] \in H^3(M;\mathbb{R})$ a non–trivial mixed class. Then the subspaces*

$$\mathcal{V}_{2,1} \subset \mathcal{H}^2(M_1) \otimes \mathcal{H}^1(M_2), \tag{2.7}$$

$$\mathcal{V}_{1,2} \subset \mathcal{H}^1(M_1) \otimes \mathcal{H}^2(M_2) \tag{2.8}$$

spanned by any minimal harmonic decomposition of the mixed Künneth components of $[\omega]$ are intrinsically defined*—i.e. they depend only on $[\omega]$ and not on the particular minimal decomposition chosen.*

Proof. For a compact oriented surface F^2 we have $b_2(F) = 1$ and $b_1(F) = 2g(F)$. Let vol_F be the unique (up to scale) g_F–harmonic 2–form. Then

$$\mathcal{H}^2(F) = \mathbb{R} \cdot \mathrm{vol}_F .$$

Consequently any non–zero $T \in \mathcal{H}^2(M_1) \otimes \mathcal{H}^1(M_2)$ has rank one and can be written uniquely as

$$T = \mathrm{vol}_{M_1} \otimes \beta, \qquad \beta \in \mathcal{H}^1(M_2).$$

Indeed, if $\mathrm{vol}_{M_1} \otimes \beta = \mathrm{vol}_{M_1} \otimes \beta'$, then $\mathrm{vol}_{M_1} \otimes (\beta - \beta') = 0$ and, because vol_{M_1} is non–zero as an element of $\mathcal{H}^2(M_1)$, we must have $\beta = \beta'$. Thus the one–dimensional subspace $\mathrm{span}\{\beta\}$ is uniquely determined by T, and

$$\mathcal{V}_{2,1} = \mathcal{H}^2(M_1) \otimes \mathrm{span}\{\beta\}$$

is intrinsically defined. The same argument applied symmetrically yields the intrinsic definition of $\mathcal{V}_{1,2}$. □

Remark 2.4 (Failure of invariance in general)**.** Invariance fails without the admissibility hypothesis. For instance, on $M = T^3 \times T^3$ we have $b_2(T^3) = 3$ and $b_1(T^3) = 3$. The mixed class

$$[\omega] = (dx \wedge dy) \otimes dz + (dx \wedge dz) \otimes dy \in H^2(T^3) \otimes H^1(T^3)$$

has rank 2. It admits two distinct minimal decompositions

$$\begin{aligned} \omega_h^{2,1} &= (dx \wedge dy) \otimes dz + (dx \wedge dz) \otimes dy \\ &= \tfrac{1}{2}\big[(dx \wedge dy + dx \wedge dz) \otimes (dz + dy) \\ &\qquad + (dx \wedge dy - dx \wedge dz) \otimes (dz - dy)\big], \end{aligned}$$

which generate different subspaces $\mathcal{V}_{2,1}$ and $\mathcal{V}'_{2,1}$. Consequently the integer $\dim\big(\mathcal{V}_{2,1} \cap (\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2))\big)$ depends on the choice, contradicting the metric–invariance.

2.2. **Mixed obstruction spaces.** Having established the intrinsic nature of the mixed factor spaces in Lemma 2.3 for admissible product manifolds (Definition 2.2), we now define the obstruction kernels.

Definition 2.5 (Mixed obstruction spaces)**.** For an admissible product manifold M and a mixed class $[\omega]$ we define

$$\mathcal{V}_{2,1} := \operatorname{span}\{\alpha_i \otimes \beta_i\} \subset \mathcal{H}^2(M_1) \otimes \mathcal{H}^1(M_2),$$
$$\mathcal{V}_{1,2} := \operatorname{span}\{\tilde{\alpha}_j \otimes \tilde{\beta}_j\} \subset \mathcal{H}^1(M_1) \otimes \mathcal{H}^2(M_2),$$

where the spans are taken over any minimal harmonic decomposition of the mixed Künneth components of $[\omega]$; by Lemma 2.3 these spans are independent of the particular choice.

The *mixed obstruction spaces* are the intersections

$$\ker \Psi_p := \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big),$$
$$\ker \widetilde{\Psi}_p := \mathcal{V}_{1,2} \cap \big(\mathcal{P}_1(M_1) \otimes \mathcal{P}_2(M_2)\big),$$

and we set $\mathcal{K} := \ker \Psi_p \oplus \ker \widetilde{\Psi}_p$.

Remark 2.6 (Anticipatory notation)**.** The notation "$\ker \Psi_p$" is anticipatory: in the proof of Theorem 5.2, we construct obstruction maps $\Psi_p : V_{2,1} \to \mathfrak{so}(T_pM)$ and $\widetilde{\Psi}_p : V_{1,2} \to \mathfrak{so}(T_pM)$ via covariant derivatives of the torsion form. A key step in the proof is to verify that the kernels of these maps coincide with the intersections defined above. The induced quotient maps $V_{2,1}/\ker \Psi_p \to \mathfrak{so}(T_pM)$ are then shown to be injective on a dense open subset, establishing the lower bound.

3. Off-Diagonal Curvature and Holonomy Complexity

Let $P_i : T_pM \to V_i$ be the orthogonal projections of the product splitting.

Definition 3.1 (Off-diagonal curvature)**.** For a curvature tensor R, define its off-diagonal component by

$$R_{\text{off}}(X,Y)Z := P_1 R(X,Y)(P_2 Z) + P_2 R(X,Y)(P_1 Z). \tag{3.1}$$

Definition 3.2 (Off-diagonal holonomy component)**.** Let (M,g) be a Riemannian product manifold with orthogonal splitting $T_pM = V_1 \oplus V_2$, and let ∇^C be a metric connection on TM with totally skew-symmetric torsion.

Let $P_i : T_pM \to V_i$ be the orthogonal projections and let $\Pi_{\text{off}} : \mathfrak{so}(T_pM) \to \mathfrak{so}(T_pM)$ be the linear projection onto off-diagonal endomorphisms:

$$\Pi_{\text{off}}(A)(Z) := P_1 A(P_2 Z) + P_2 A(P_1 Z).$$

Define the space of off-diagonal skew-symmetric endomorphisms as the image of this projection:

$$\mathfrak{so}^{\mathrm{off}}(T_pM) := \mathrm{Im}(\Pi_{\mathrm{off}}) = \{A \in \mathfrak{so}(T_pM) \mid A(V_1) \subset V_2,\ A(V_2) \subset V_1\}.$$

The *off-diagonal holonomy component* is the intersection of the holonomy algebra with this space:

$$\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) := \mathfrak{hol}_p(\nabla^C) \cap \mathfrak{so}^{\mathrm{off}}(T_pM).$$

Equivalently, $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) = \{A \in \mathfrak{hol}_p(\nabla^C) \mid \Pi_{\mathrm{off}}(A) = A\}$ consists of those holonomy elements that are purely off-diagonal.

Remark 3.3. Although $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C)$ is a vector subspace of $\mathfrak{hol}_p(\nabla^C)$, it is not closed under the Lie bracket, since $[\mathfrak{so}^{\mathrm{off}}, \mathfrak{so}^{\mathrm{off}}] \subset \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$. Nevertheless, $\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) > 0$ implies that the holonomy representation is irreducible with respect to the splitting $V_1 \oplus V_2$, therefore, we have global irreducibility.

By the Ambrose–Singer theorem [3], $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C)$ depends only on the global holonomy algebra of ∇^C and is therefore independent of the choice of local generators and of the pointwise rank of the curvature tensor. In particular, at each point $p \in M$ one has, clearly, $\dim \mathfrak{hol}_p(\nabla^C) \geq \dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C)$. Because ∇^C is metric, its holonomy algebra sits in $\mathfrak{so}(T_pM)$. If the action were reducible with respect to $V_1 \oplus V_2$, all holonomy endomorphisms would be block-diagonal, forcing $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) = \{0\}$. Hence nontrivial off-diagonal holonomy implies irreducibility.

4. Mixed Rank of a Degree-3 Class

Write the Künneth decomposition of $H^3(M)$ as:

$$\begin{gathered}\big(H^3(M_1) \otimes H^0(M_2)\big) \ \oplus\ \big(H^2(M_1) \otimes H^1(M_2)\big) \oplus \\ \big(H^1(M_1) \otimes H^2(M_2)\big) \ \oplus\ \big(H^0(M_1) \otimes H^3(M_2)\big).\end{gathered} \tag{4.1}$$

Mixed components occur precisely in bidegrees $(2,1)$ and $(1,2)$.

Definition 4.1 (Mixed tensor rank)**.** Let $[\omega] \in H^3(M;\mathbb{R})$. Denote by $\pi_{2,1}([\omega]) \in H^2(M_1) \otimes H^1(M_2)$ and $\pi_{1,2}([\omega]) \in H^1(M_1) \otimes H^2(M_2)$ its mixed Künneth projections. Define

$$r_{2,1} = \min\Big\{r \ :\ \pi_{2,1}([\omega]) = \sum_{i=1}^{r} \alpha_i \otimes \beta_i,\ \alpha_i \in H^2(M_1),\ \beta_i \in H^1(M_2)\Big\}, \tag{4.2}$$

and analogously $r_{1,2}$ for $\pi_{1,2}([\omega])$. The mixed rank is

$$\mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) := r_{2,1} + r_{1,2}. \tag{4.3}$$

For degree 3, the tensor rank in each bidegree is well-defined and finite, and can be computed from any decomposition of the harmonic representative ω_h into wedge products of harmonic forms on the factors for the product metric g. Although ω_h depends on g, the ranks $r_{2,1}, r_{1,2}$ are purely topological (indeed, the tensor rank of an element in $H^2(M_1) \otimes H^1(M_2)$ is defined as the minimal number of simple tensors in any decomposition, hence is invariant under change of metric within the product class).

5. Main Result and Proof

Lemma 5.1 (Existence of points of maximal rank for covariant derivatives)**.** *Let (M_1, g_1) be a compact oriented Riemannian manifold and let $\mathcal{H}^2(M_1, g_1)$ denote the space of harmonic 2-forms. Given a subspace $U \subset \mathcal{H}^2(M_1, g_1)$ of dimension r and its parallel part $U_\parallel := U \cap \mathcal{P}_2(M_1, g_1)$ of dimension s, choose a basis $\alpha_1, \dots, \alpha_r$ of U such that $\alpha_1, \dots, \alpha_s$ span $U_\parallel$.*

If $r - s \leq 1$ (condition satisfied by admissible product manifolds: U is the span of a minimal harmonic decomposition, so $\dim U = r_{2,1} \leq 1$, hence $r - s \leq 1$; see Definition 4.1, Definition 2.2 and Lemma 2.3), then there exists a dense open set $\mathcal{U} \subset M_1$ on which the covariant derivatives

$$\left(\nabla^{g_1} \alpha_{s+1}, \dots, \nabla^{g_1} \alpha_r\right)$$

are pointwise linearly independent; i.e.

$$\mathcal{U} = \left\{ p \in M_1 \;\middle|\; \operatorname{rank}\left[(\nabla^{g_1} \alpha_i)(p)\right]_{i=s+1,\dots,r} = r - s \right\} \neq \emptyset.$$

Proof. We proceed in three steps.
1. Reduction to a real-analytic metric. Real-analytic Riemannian metrics are dense in the space of smooth metrics on a compact manifold; this follows from the theorem of Greene–Jacobowitz on analytic approximation of smooth maps (see [5]). Hence we can choose a sequence of real-analytic metrics $\{g_1^{(k)}\}_{k \in \mathbb{N}}$ converging to g_1 in the C^∞ topology. The dimension of the space of parallel 2-forms is upper-semicontinuous with respect to the metric (see [4]); therefore, after passing to a subsequence, we may assume

$$\dim \mathcal{P}_2\big(M_1, g_1^{(k)}\big) \leq \dim \mathcal{P}_2(M_1, g_1) = s \qquad \text{for all } k.$$

For each k let $\alpha_i^{(k)} \in \mathcal{H}^2\big(M_1, g_1^{(k)}\big)$ be the unique harmonic representative of the cohomology class $[\alpha_i]$. By the stability theorem for harmonic forms under metric deformation (Warner [10], see also Kodaira–Spencer [8]), $\alpha_i^{(k)} \to \alpha_i$ in C^∞. Consequently the covariant derivatives converge in C^1:

$$\nabla^{g_1^{(k)}} \alpha_i^{(k)} \longrightarrow \nabla^{g_1} \alpha_i \qquad (i = s+1, \dots, r).$$

If we prove that for each k there exists a dense open set $\mathcal{U}^{(k)} \subset M_1$ where the $\nabla^{g_1^{(k)}} \alpha_i^{(k)}$ are linearly independent, then the C^1 convergence guarantees that, for large enough k, the same linear independence holds on a dense open set for the original metric g_1. Indeed, linear independence is equivalent to the non-vanishing of at least one $(r-s) \times (r-s)$ minor of the matrix of components; each minor depends continuously on the metric and the forms, so a minor that is non-zero on an open set for $g_1^{(k)}$ remains non-zero on an open set for g_1 when k is sufficiently large.

Thus it suffices to prove the lemma under the assumption that g_1 is real-analytic.

2. Admissible manifolds. Assume now that g_1 is real-analytic. Then every harmonic form is real-analytic (by elliptic regularity for the Hodge–Laplacian, see [9]), and consequently the covariant derivatives $\nabla^{g_1}\alpha_i$ are real-analytic sections of the vector bundle $E := T^*M_1 \otimes \Lambda^2 T^*M_1$.

Consider the real-analytic section

$$\Phi : M_1 \longrightarrow \Lambda^{r-s}E, \qquad \Phi(p) := (\nabla^{g_1}\alpha_{s+1})(p) \wedge \cdots \wedge (\nabla^{g_1}\alpha_r)(p).$$

The set $\mathcal{U} := \{p \in M_1 \mid \Phi(p) \neq 0\}$ is exactly the set where the covariant derivatives are linearly independent; because Φ is continuous, $\mathcal{U}$ is open. We must show that $\mathcal{U} \neq \emptyset$.

Suppose, for contradiction, that $\mathcal{U} = \emptyset$; i.e. $\Phi(p) = 0$ for every $p \in M_1$. Then at each point the sections $\nabla^{g_1}\alpha_{s+1}, \dots, \nabla^{g_1}\alpha_r$ are linearly dependent. Since the condition of rank $\leq d$ is given by the vanishing of all $(d+1)\times(d+1)$ minors, which are analytic functions, the locus where the rank is constant is an analytic subset. Let $d < r - s$ be the minimal rank and let p_0 be a point where the rank equals d. There exists a connected open neighbourhood V of p_0 on which the rank is constant and equal to d.

On V we can therefore find real-analytic functions $c_1, \dots, c_{r-s} : V \to \mathbb{R}$, not all zero, such that

$$\sum_{i=1}^{r-s} c_i(p)\big(\nabla^{g_1}\alpha_{s+i}\big)(p) = 0 \qquad \forall p \in V. \tag{5.1}$$

Since we have to consider admissible manifolds (Definition 2.2 and Lemma 2.3) we have two cases:

Case $r = s$: There are no non-parallel forms to consider; the lemma holds vacuously.

Case $r = s + 1$: There exists a unique non-parallel form α and a single analytic function c such that (5.1) reads $c(p)(\nabla^{g_1}\alpha)_p = 0$ for all $p \in V$. If $c(p_0) \neq 0$ for some $p_0 \in V$, then by analyticity $c(p) \neq 0$ on an open subset $V' \subset V$, whence $(\nabla^{g_1}\alpha)_p = 0$ on V'. By the unique-continuation principle for analytic sections (Aronszajn [1]; for differential forms see [2]), $\nabla^{g_1}\alpha \equiv 0$ on the whole connected component containing V', and hence on all of M_1. This contradicts the assumption that α is non-parallel modulo $\mathcal{P}_2(M_1, g_1)$. Therefore $c \equiv 0$ on V, contradicting the choice of the c_i's. Hence $\mathcal{U} \neq \emptyset$. Moreover, the complement $M_1 \setminus \mathcal{U} = \{p \in M_1 : (\nabla^{g_1}\alpha)(p) = 0\}$ has empty interior: if it contained an open set V', then by unique continuation $\nabla^{g_1}\alpha \equiv 0$ on all of M_1, contradicting the hypothesis that α is non-parallel. Therefore $\mathcal{U}$ is dense in M_1.

3. Returning to the original smooth metric. By Step 2, for each analytic approximation $g_1^{(k)}$, the set $\mathcal{U}^{(k)} = \{p : \nabla^{g_1^{(k)}}\alpha^{(k)}(p) \neq 0\}$ is dense in M_1.

The same conclusion holds for the smooth metric g_1 by applying the unique continuation principle for harmonic forms on smooth Riemannian manifolds [2] directly: if $M_1 \setminus \mathcal{U}$ contained an open set V, lifting to the universal cover $\tilde{M}_1$ yields a globally parallel form $\tilde{\beta}$ with $\tilde{\alpha}|_{\tilde{V}} = \tilde{\beta}|_{\tilde{V}}$. Since $\tilde{\alpha} - \tilde{\beta}$ is harmonic and vanishes on $\tilde{V}$, unique continuation [2] implies $\tilde{\alpha} = \tilde{\beta}$, so α is parallel on M_1, contradicting $\alpha \notin U_{\|}$. Thus $\mathcal{U}$ is dense. □

The proof uses only properties of harmonic forms that hold for any degree k: analytic approximation, semicontinuity of the dimension of parallel forms, C^1 convergence, and unique continuation. Therefore, the same result holds for $\mathcal{H}^1(M_1, g_1)$, as well as for $\mathcal{H}^1(M_2, g_2)$ and $\mathcal{H}^2(M_2, g_2)$.

Theorem 5.2 (Topological Lower Bound for Off-Diagonal Curvature)**.** *Let $M = M_1 \times M_2$ be an admissible compact oriented Riemannian product manifold (Definition 2.2) with product metric $g = g_1 \oplus g_2$. Let ∇^C be a cohomologically calibrated metric connection with totally skew torsion T such that $[\omega] = [\omega_h]$ is a non-trivial mixed class (Definition 2.1).*

Define the mixed factor subspaces

$$\mathcal{V}_{2,1} := \operatorname{span}\{\alpha_i \otimes \beta_i\} \subset \mathcal{H}^2(M_1) \otimes \mathcal{H}^1(M_2),$$

$$\mathcal{V}_{1,2} := \operatorname{span}\{\tilde{\alpha}_j \otimes \tilde{\beta}_j\} \subset \mathcal{H}^1(M_1) \otimes \mathcal{H}^2(M_2),$$

where the spans are taken over any minimal harmonic decomposition of the mixed Künneth components of $[\omega]$; in the admissible setting these spans are independent of the particular choice (Lemma 2.3).

Define the obstruction kernels

$$\ker \Psi_p := \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big), \quad \ker \widetilde{\Psi}_p := \mathcal{V}_{1,2} \cap \big(\mathcal{P}_1(M_1) \otimes \mathcal{P}_2(M_2)\big),$$

and set $\mathcal{K} := \ker \Psi_p \oplus \ker \widetilde{\Psi}_p$.

Then, there exists a non-empty open subset $\mathcal{V} \subset M$ such that for every $p \in \mathcal{V}$,

$$\dim\big(\mathfrak{hol}_p^{\mathrm{off}}(\nabla^C)\big) \;\geq\; r^\sharp := \min\big(\operatorname{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K},\ 1\big).$$

The bound depends only on $[\omega]_{mixed}$ and on the parallel-form strata $\mathcal{P}_k(M_i)$; hence it is invariant under any deformation of g_1 or g_2 that preserves these strata.

Proof. Let ω_h be the g-harmonic representative of $[\omega]$. Decompose ω_h according to Künneth and Hodge on the factors. For the $(2,1)$-part, write

$$\omega_h^{2,1} = \sum_{i=1}^{r_{2,1}} \alpha_i \wedge \beta_i, \tag{5.2}$$

with $\alpha_i \in \mathcal{H}^2(M_1)$ and $\beta_i \in \mathcal{H}^1(M_2)$ harmonic and chosen so that the number of terms is minimal. Similarly,

$$\omega_h^{1,2} = \sum_{j=1}^{r_{1,2}} \tilde{\alpha}_j \wedge \tilde{\beta}_j, \tag{5.3}$$

with $\tilde{\alpha}_j \in \mathcal{H}^1(M_1)$ and $\tilde{\beta}_j \in \mathcal{H}^2(M_2)$. The total number of simple mixed tensors equals $r := r_{2,1} + r_{1,2} = \operatorname{rank}_{\mathbb{R}} \big([\omega]_{\text{mixed}}\big)$.

Set $r^\sharp := \min(r - \dim \mathcal{K},\, 1)$. We will prove $\dim \mathfrak{hol}_p^{\text{off}}(\nabla^C) \geq r^\sharp$.

Consider the curvature formula (2.5) for $X \in V_1$ and $Y \in V_2$:

$$R^C(X,Y) \;=\; R^{LC}(X,Y) + \tfrac{1}{2}\big[(\nabla^{LC}_X T)(Y,\cdot,\cdot)^\sharp - (\nabla^{LC}_Y T)(X,\cdot,\cdot)^\sharp\big] + \tfrac{1}{4}\, Q_T(X,Y).$$

On a Riemannian product the mixed Riemann curvature vanishes: $R^{LC}(X,Y) = 0$ whenever $X \in V_1$ and $Y \in V_2$ (since $\nabla^{LC}_X Y = 0 = [X,Y]$ for vector fields tangent to different factors). For the remaining three terms we exploit the rigid pointwise algebraic structure that the surface hypothesis imposes on mixed 3-forms.

Since both factors are compact oriented surfaces, $\dim V_i = 2$, and the exterior power $\Lambda^2 V_1^*$ is one-dimensional, spanned by the area form vol_{V_1}. Every 3-form of bidegree $(2,1)$ on M can therefore be written at each point p as

$$T^{2,1}_p \;=\; h(p)\ \mathrm{vol}_{V_1} \wedge \tau_p, \tag{5.4}$$

where $h\colon M \to \mathbb{R}$ is a smooth function and τ is a smooth section of V_2^* (depending on the full point $p = (p_1, p_2)$). Likewise, every $(1,2)$-form satisfies $T^{1,2}_p = \widetilde{h}(p)\, \widetilde{\sigma}_p \wedge \mathrm{vol}_{V_2}$ with $\widetilde{\sigma} \in \Gamma(V_1^*)$. These representations hold for the *full* torsion T, not merely for its harmonic projection ω_h.

We denote by $J_i\colon V_i \to V_i$ the volume endomorphism of the oriented surface M_i, characterised by $g(J_i u, v) = \mathrm{vol}_{V_i}(u,v)$; it is a skew-symmetric isometry with $J_i^2 = -\mathrm{Id}$, and in particular $J_i X \perp X$ for every nonzero $X \in V_i$. Let $\pi_0\colon \mathfrak{so}(T_pM) \to \mathfrak{g}_0 := \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$ and $\pi_1\colon \mathfrak{so}(T_pM) \to \mathfrak{g}_1 := \mathfrak{so}^{\text{off}}(T_pM)$ denote the projections induced by the $\mathbb{Z}_2$-grading of $\mathfrak{so}(T_pM)$.

A direct computation using (5.4) yields the following structure for the three surviving terms. The first, $\frac{1}{2}(\nabla^{LC}_X T)(Y,\cdot,\cdot)^\sharp$, is purely diagonal: applying the Leibniz rule to $\nabla^{LC}_X(h\ \mathrm{vol}_{V_1} \wedge \tau)$ and using $\nabla^{LC}_X \mathrm{vol}_{V_1} = 0$ (since $X \in V_1$ and the volume form of M_1 is parallel), one finds that this term is proportional to the volume endomorphism J_1 on V_1 (respectively J_2 on V_2 for the $(1,2)$-component of T); thus

$$\pi_0\big(R^C(X,Y)\big) \;=\; \lambda(X,Y)\, J_1 \;+\; \mu(X,Y)\, J_2 \tag{5.5}$$

for certain scalars λ, μ. The second, $-\frac{1}{2}(\nabla^{LC}_Y T)(X,\cdot,\cdot)^\sharp$, is purely off-diagonal. Defining the 1-form $\rho_Y \in V_2^*$ by

$$\rho_Y \;:=\; (Yh)\,\tau \;+\; h\,\nabla^{g_2}_Y \tau \;=\; \nabla^{g_2}_Y(h\tau), \tag{5.6}$$

the restriction to $\mathrm{Hom}(V_1, V_2)$ reads

$$-\tfrac{1}{2}(\nabla^{LC}_Y T^{2,1})(X,Z,\cdot)^\sharp\Big|_{Z \in V_1} \;=\; -\tfrac{1}{2}\, g(J_1 X, Z)\ \rho_Y^\sharp \ \in V_2. \tag{5.7}$$

The third, $\frac{1}{4} Q_T(X,Y)$, is also purely off-diagonal (by a bidegree computation), with restriction

$$\tfrac{1}{4}\, Q_T(X,Y)\big|_{Z \in V_1} \;=\; \tfrac{h^2}{4}\ \tau(Y)\ g(X,Z)\ \tau^\sharp \ \in V_2. \tag{5.8}$$

To derive 5.8, observe that the contortion tensor $K_X Y = \frac{1}{2} T(X, Y, \cdot)^\sharp$ acts on $Z \in V_1$ via the $(2,1)$-component as

$$K_Y^{(2,1)} Z = \frac{h}{2}\,\tau(Y)\,J_1 Z,$$

since $(\mathrm{vol}_{V_1} \wedge \tau)(Y, Z, W) = \tau(Y)\,\mathrm{vol}_{V_1}(Z, W)$ for $Y \in V_2$ and $Z, W \in V_1$. The iterated contortion $K_X(K_Y Z)$ with $X \in V_1$ then gives

$$K_X(K_Y Z) = \frac{h^2}{4}\,\tau(Y)\,g(X, Z)\,\tau^\sharp,$$

where we used $T^{(2,1)}(X, J_1 Z, W) = h\ \mathrm{vol}_{V_1}(X, J_1 Z)\,\tau(W) = h\,g(J_1 X, J_1 Z)\,\tau(W) = h\,g(X, Z)\,\tau(W)$, the last equality following from J_1 being an isometry. Conversely, $K_Y(K_X Z) = 0$: since $K_X Z = \frac{h}{2}\,\mathrm{vol}_{V_1}(X, Z)\,\tau^\sharp \in V_2$ and $T^{(2,1)}(Y, W, \cdot) = 0$ whenever $Y, W \in V_2$ (the form vol_{V_1} requires two arguments in V_1). Therefore $Q_T(X, Y)Z = K_X(K_Y Z) - K_Y(K_X Z)$ yields 5.8.

Combining (5.7) and (5.8), the full off-diagonal projection restricted to $\mathrm{Hom}(V_1, V_2)$ is

$$\pi_1\big(R^C(X, Y)\big)\big|_{V_1 \to V_2}\ :\ Z\ \longmapsto\ -\tfrac{1}{2}\,g(J_1 X, Z)\,\rho_Y^\sharp\ +\ \tfrac{h^2}{4}\,\tau(Y)\,g(X, Z)\,\tau^\sharp. \tag{5.9}$$

The two summands act through the functionals $g(J_1 X, \cdot\,)$ and $g(X, \cdot\,)$ on V_1 respectively. Because $J_1 X \perp X$, these functionals are orthogonal, and the two contributions cannot cancel: if the map (5.9) vanishes for all $Z \in V_1$, then

$$\rho_Y^\sharp = 0 \qquad \text{and} \qquad h^2\,\tau(Y)\,\tau^\sharp = 0 \tag{5.10}$$

must hold separately.

We now observe that the quadratic term (5.8) is non-vanishing on a non-empty open set. If $r^\sharp_{2,1} = 1$, the harmonic projection $\omega_h^{2,1} = c \cdot \mathrm{vol}_{M_1} \wedge \beta$ is non-zero, so $[T^{2,1}] = [\omega_h^{2,1}] \neq 0$ in cohomology. Since $T^{2,1} = \omega_h^{2,1} + \eta^{2,1}$ with $\eta^{2,1}$ exact plus coexact, Hodge orthogonality gives $T^{2,1} \not\equiv 0$ on M (otherwise $\omega_h^{2,1} = -\eta^{2,1}$ would be simultaneously harmonic and exact-plus-coexact, forcing $\omega_h^{2,1} = 0$). The set

$$\mathcal{W}^{2,1}\ :=\ \{p \in M : T_p^{2,1} \neq 0\} \tag{5.11}$$

is therefore a non-empty open subset of M. On $\mathcal{W}^{2,1}$ we have $h(p) \neq 0$ and $\tau_p \neq 0$ (by (5.4)), so choosing $Y \in V_2$ with $\tau_p(Y) \neq 0$ yields $h^2 \tau(Y) \tau^\sharp \neq 0$; the second condition in (5.10) is therefore violated, and $\pi_1(R^C(X, Y)) \neq 0$. The symmetric argument for the $(1,2)$-component (using $J_2 Y \perp Y$) produces a non-empty open set $\mathcal{W}^{1,2}$ whenever $r^\sharp_{1,2} = 1$.

In particular, the exact-plus-coexact part η and the quadratic torsion term Q_T cannot reduce the number of independent off-diagonal curvature directions below the topological invariant $r^\sharp$. To compute this invariant explicitly and to construct $r^\sharp$ independent off-diagonal operators, it therefore suffices to work with the harmonic projection ω_h alone, as we now proceed to do.

To organise these contributions, choose the harmonic forms $\{\alpha_i\}$ on M_1 and $\{\beta_i\}$ on M_2 mutually L^2-orthonormal within their degrees, and likewise for $\{\tilde{\alpha}_j\}$, $\{\tilde{\beta}_j\}$. By minimality of the tensor-rank decomposition, the sets $\{\alpha_i\}$ and $\{\beta_i\}$ are linearly independent as sections; hence there exists a dense open subset of points $p = (p_1, p_2) \in M_1 \times M_2$ where their evaluations $\{\alpha_i|_{p_1}\} \subset \Lambda^2 T^*_{p_1} M_1$ and $\{\beta_i|_{p_2}\} \subset T^*_{p_2} M_2$ are linearly independent. For any such p we pick vectors $X_i, Y_i \in T_{p_1} M_1$ and $Z_i \in T_{p_2} M_2$ satisfying $\alpha_i(X_i, Y_i)|_p = 1$ and $\beta_i(Z_i)|_p = 1$.

Define the linear map

$$\Psi_p : \mathcal{V}_{2,1} \longrightarrow \mathfrak{so}(T_p M), \qquad \Psi_p\Big(\sum_{i=1}^{r_{2,1}} c_i \alpha_i \otimes \beta_i\Big) := \frac{1}{2} \sum_{i=1}^{r_{2,1}} c_i \, (\nabla^{LC} \omega_h)^{\text{off}} (X_i, Y_i, Z_i), \tag{5.12}$$

where the vectors $X_i, Y_i \in T_{p_1} M_1$ and $Z_i \in T_{p_2} M_2$ satisfy $\alpha_i(X_i, Y_i) = 1$ and $\beta_i(Z_i) = 1$. The notation on the right-hand side denotes the off-diagonal component of the curvature contribution from the $\nabla^{LC}\omega_h$-term in 2.5, and its precise meaning is established through the kernel characterization in *Claim 5.2.1*.

Claim 5.2.1 (Independence of choices). The subspace $\ker \Psi_p \subset \mathcal{V}_{2,1}$ is independent of the choice of normalizing vectors $\{X_i, Y_i, Z_i\}$ in 5.12.

Proof of Claim. Writing $\omega_h^{2,1} = \sum_{i=1}^{r_{2,1}} \alpha_i \wedge \beta_i$, the product rule gives

$$(\nabla^{LC} \omega_h)^{\text{off}} = \sum_{i=1}^{r_{2,1}} (\nabla^{g_1} \alpha_i) \wedge \beta_i + \alpha_i \wedge (\nabla^{g_2} \beta_i).$$

For vectors $X_i, Y_i \in T_{p_1} M_1$, $Z_i \in T_{p_2} M_2$ with $\alpha_i(X_i, Y_i) = 1$, $\beta_i(Z_i) = 1$, the contraction $\iota_{Z_i} \iota_{Y_i} \iota_{X_i}$ extracts precisely the i-th component:

$$(\nabla^{LC} \omega_h)^{\text{off}} (X_i, Y_i; Z_i) = (\nabla^{g_1} \alpha_i) \wedge \beta_i + \alpha_i \wedge (\nabla^{g_2} \beta_i).$$

Summing over i with coefficients c_i yields that $\Psi_p(\xi) = 0$ is equivalent to the tensor equation

$$\sum_{i=1}^{r_{2,1}} c_i \nabla^{LC} (\alpha_i \wedge \beta_i) = 0, \tag{5.13}$$

which is manifestly independent of the vectors $\{X_i, Y_i, Z_i\}$. Hence $\ker \Psi_p$ is well-defined. □

Now, although the right-hand side of 5.12 depends on the choice of vectors, by *Claim 5.2.1* the condition $\Psi_p(\xi) = 0$ is independent of that choice.

If $\xi \in \ker \Psi_p$, then by definition

$$\frac{1}{2} \sum_i c_i (\nabla^{LC} \omega_h)^{\text{off}} (X_i, Y_i, Z_i) = 0.$$

Separating the bidegrees in 5.13 yields the two independent conditions

$$\sum_{i=1}^{r_{2,1}} c_i(\nabla^{g_1}\alpha_i) \wedge \beta_i = 0, \quad \sum_{i=1}^{r_{2,1}} c_i\alpha_i \wedge (\nabla^{g_2}\beta_i) = 0. \tag{5.14}$$

Since the harmonic forms β_i are linearly independent in $\mathcal{H}^1(M_2)$, the set of points where they are pointwise linearly independent is a dense open subset of M_2 (its complement is the zero locus of certain smooth determinants). Denote this open set by $\mathcal{U}_2^{2,1} \subset M_2$.

Now fix any point $y_0 \in \mathcal{U}_2^{2,1}$. For this fixed y_0, the first equation in (5.14) becomes:

$$\Big(\sum_{i=1}^{r_{2,1}} c_i \nabla^{g_1}\alpha_i(x)\Big) \wedge \beta_i(y_0) = 0 \quad \text{for all } x \in M_1.$$

Since the covectors $\beta_1(y_0), \dots, \beta_{r_{2,1}}(y_0)$ are linearly independent, this forces

$$\sum_{i=1}^{r_{2,1}} c_i \nabla^{g_1}\alpha_i(x) = 0 \quad \text{for all } x \in M_1.$$

Analogously, because the harmonic forms α_i are linearly independent in $\mathcal{H}^2(M_1)$, there exists a dense open set $\mathcal{U}_1^{2,1} \subset M_1$ where they are pointwise linearly independent.

Fix $x_0 \in \mathcal{U}_1^{2,1}$. The second equation in (5.14) then gives for all $y \in M_2$:

$$\Big(\sum_{i=1}^{r_{2,1}} c_i\alpha_i(x_0)\Big) \wedge \nabla^{g_2}\beta_i(y) = 0,$$

which implies $\sum_{i=1}^{r_{2,1}} c_i \nabla^{g_2}\beta_i(y) = 0$ for all $y \in M_2$.

Therefore,

$$\eta_1 := \sum_{i=1}^{r_{2,1}} c_i\alpha_i \in \mathcal{P}_2(M_1), \qquad \eta_2 := \sum_{i=1}^{r_{2,1}} c_i\beta_i \in \mathcal{P}_1(M_2). \tag{5.15}$$

(Note that the coefficients $\{c_i\}$ are the same in both conditions, being the components of the fixed element $\xi = \sum c_i\alpha_i \otimes \beta_i$ in the tensor product decomposition.)

Conversely, if (5.15) holds, then $\nabla^{LC}\big(\sum_i c_i\alpha_i \wedge \beta_i\big) = 0$, which forces $\Psi_p\big(\sum_i c_i\alpha_i \otimes \beta_i\big) = 0$. Therefore $\xi = \sum_i c_i\alpha_i \otimes \beta_i$ belongs to $\ker \Psi_p$ if and only if (5.15) is satisfied.

Now we identify this kernel with the intersection $\mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big)$. If $\xi \in \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2 \otimes \mathcal{P}_1\big)$, then ξ can be written as a finite sum $\sum_k A_k \otimes B_k$ with $A_k \in \mathcal{P}_2$, $B_k \in \mathcal{P}_1$. For each term, $\nabla^{LC}(A_k \wedge B_k) = 0$ because A_k and B_k are parallel, hence $\Psi_p(A_k \otimes B_k) = 0$. By linearity, $\Psi_p(\xi) = 0$, so $\xi \in \ker \Psi_p$.

For the converse, assume that $\xi = \sum_{i=1}^{r_{2,1}} c_i\alpha_i \otimes \beta_i \in \ker \Psi_p$. Then (5.15) holds, i.e.

$$\eta_1 = \sum_{i=1}^{r_{2,1}} c_i\alpha_i \in \mathcal{P}_2(M_1), \qquad \eta_2 = \sum_{i=1}^{r_{2,1}} c_i\beta_i \in \mathcal{P}_1(M_2).$$

Let $U = \operatorname{span}\{\alpha_i\} \subset \mathcal{H}^2(M_1)$ and $V = \operatorname{span}\{\beta_i\} \subset \mathcal{H}^1(M_2)$. Define

$$U_\| = U \cap \mathcal{P}_2(M_1), \qquad V_\| = V \cap \mathcal{P}_1(M_2),$$

and set

$$s = \dim U_\|, \qquad t = \dim V_\|.$$

Choose any basis $\{\alpha_1, \dots, \alpha_s\}$ of $U_\|$ and complete it to a basis of U by adding vectors $\{\alpha_{s+1}, \dots, \alpha_{r_{2,1}}\}$ whose classes form a basis of $U/U_\|$ (hence linearly independent modulo $\mathcal{P}_2(M_1)$).

Now apply Lemma 5.1 to the subspace $U \subset \mathcal{H}^2(M_1)$ with its parallel part $U_\|$ (which has dimension s). The lemma guarantees a dense open set $\mathcal{U}_1^{2,1} \subset M_1$ such that for every $p_1 \in \mathcal{U}_1^{2,1}$, the covariant derivatives $\{\nabla^{g_1}\alpha_i(p_1)\}_{i=s+1}^{r_{2,1}}$ are linearly independent.

Since η_1 is parallel, $\nabla^{g_1}\eta_1 \equiv 0$; evaluating at p_1 we obtain

$$\sum_{i=s+1}^{r_{2,1}} c_i \nabla^{g_1}\alpha_i(p_1) = 0.$$

The linear independence at p_1 forces $c_i = 0$ for all $i = s+1, \dots, r_{2,1}$.

An identical argument applied to M_2 yields a point $p_2 \in \mathcal{U}_2^{2,1}$ such that

$$\sum_{i=t+1}^{r_{2,1}} c_i \nabla^{g_2}\beta_i(p_2) = 0,$$

and the linear independence of $\{\nabla^{g_2}\beta_i(p_2)\}_{i=t+1}^{r_{2,1}}$ forces $c_i = 0$ for all $i = t+1, \dots, r_{2,1}$.

Hence, the only possibly non-zero coefficients are those with $i \le \min(s,t)$. For such indices, by construction $\alpha_i \in \mathcal{P}_2(M_1)$ and $\beta_i \in \mathcal{P}_1(M_2)$. Therefore,

$$\xi = \sum_{i=1}^{\min(s,t)} c_i\alpha_i \otimes \beta_i \in \mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2).$$

Since $\xi \in \mathcal{V}_{2,1}$ by hypothesis, we have shown

$$\ker \Psi_p \subseteq \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big).$$

The opposite inclusion has already been proved above. Thus

$$\ker \Psi_p = \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big).$$

An identical argument applied to the $(1,2)$-component yields

$$\ker \widetilde{\Psi}_p = \mathcal{V}_{1,2} \cap \big(\mathcal{P}_1(M_1) \otimes \mathcal{P}_2(M_2)\big).$$

Now consider the induced map on the quotient

$$\overline{\Psi}_p : \mathcal{V}_{2,1}/\ker\Psi_p \longrightarrow \mathfrak{so}(T_pM), \qquad \overline{\Psi}_p([\xi]) := \Psi_p(\xi).$$

The map $\overline{\Psi}_p$ is well-defined and linear. It is injective because $\overline{\Psi}_p([\xi]) = 0$ implies $\Psi_p(\xi) = 0$, hence $\xi \in \ker\Psi_p$ and $[\xi] = 0$. Consequently, $\overline{\Psi}_p$ sends a basis of the quotient to a set of linearly independent off-diagonal curvature operators, and we obtain

$$\dim(\operatorname{im}\Psi_p) = \dim\big(\mathcal{V}_{2,1}/\ker\Psi_p\big) = r_{2,1} - \dim\ker\Psi_p. \tag{5.16}$$

An entirely analogous argument for the $(1,2)$-component gives

$$\dim(\operatorname{im}\widetilde{\Psi}_p) = r_{1,2} - \dim\ker\widetilde{\Psi}_p. \tag{5.17}$$

Recall that $s = \dim(U_\|)$ and $t = \dim(V_\|)$. By Lemma 5.1 applied to the subspace $\operatorname{span}\{\alpha_i\} \subset \mathcal{H}^2(M_1)$ with parallel part of dimension s, there exists a dense open set $\mathcal{U}_1^{2,1} \subset M_1$ such that for every $p_1 \in \mathcal{U}_1^{2,1}$ the covariant derivatives $\nabla^{g_1}\alpha_i(p_1)$ (for $i = s+1, \dots, r_{2,1}$) are linearly independent. Similarly, applying Lemma 5.1 to $\operatorname{span}\{\beta_i\} \subset \mathcal{H}^1(M_2)$ with parallel part of dimension t yields a dense open set $\mathcal{U}_2^{2,1} \subset M_2$ such that for every $p_2 \in \mathcal{U}_2^{2,1}$ the covariant derivatives $\nabla^{g_2}\beta_i(p_2)$ (for $i = t+1, \dots, r_{2,1}$) are linearly independent.

By a symmetric argument applied to the (1,2)-part, we obtain dense open sets for the corresponding forms on each factor. By Lemma 5.1, each of these open sets is dense (its complement has empty interior by unique continuation). Define

$$\mathcal{U}_1 := \mathcal{U}_1^{(2,1)} \cap \mathcal{U}_1^{(1,2)} \subset M_1, \qquad \mathcal{U}_2 := \mathcal{U}_2^{(2,1)} \cap \mathcal{U}_2^{(1,2)} \subset M_2,$$

where $\mathcal{U}_1^{(2,1)}$ and $\mathcal{U}_1^{(1,2)}$ are the dense open sets on M_1 arising from the $(2,1)$ and $(1,2)$ components respectively, and similarly for M_2. Since M_1 and M_2 are compact (hence complete) metric spaces, finite intersections of dense open sets are dense, hence $\mathcal{U}_1$ and $\mathcal{U}_2$ are dense open subsets of M_1 and M_2 respectively.

Consequently, the product set $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$ is a dense open subset of $M = M_1 \times M_2$. For any point $p = (p_1, p_2) \in \mathcal{U}$ the linear maps Ψ_p and $\widetilde{\Psi}_p$ have maximal rank, i.e.

$$\dim\operatorname{im}\Psi_p = r_{2,1} - \dim\ker\Psi_p, \qquad \dim\operatorname{im}\widetilde{\Psi}_p = r_{1,2} - \dim\ker\widetilde{\Psi}_p.$$

Hence, at every $p \in \mathcal{U}$, the off-diagonal part of the linear term $\frac{1}{2}(\nabla^{LC}T)_{\text{off}}$ generates at least

$$\big(r_{2,1} - \dim\ker\Psi_p\big) + \big(r_{1,2} - \dim\ker\widetilde{\Psi}_p\big) = r - \dim\mathcal{K}$$

linearly independent curvature operators (each having non-trivial off-diagonal projection).

It remains to produce $r^\sharp$ linearly independent *purely* off-diagonal elements of the holonomy algebra. Since in the admissible setting $r^\sharp_{2,1} \leq 1$ and $r^\sharp_{1,2} \leq 1$, we have $r - \dim\mathcal{K} \leq 2$; the value $r - \dim\mathcal{K} = 2$ occurs only when

$r^\sharp_{2,1} = r^\sharp_{1,2} = 1$. In that case, constructing two linearly independent purely off-diagonal elements at a single point would require $p \in \mathcal{W}^{2,1} \cap \mathcal{W}^{1,2}$, but the bidegree components $T^{2,1}$ and $T^{1,2}$ may have disjoint pointwise support, so this intersection can be empty. However, the sets $\mathcal{W}^{2,1} \cap \mathcal{U}$ and $\mathcal{W}^{1,2} \cap \mathcal{U}$ are individually non-empty open subsets of M (each is a non-empty open set intersected with a dense open set), and on each one the construction below produces at least one purely off-diagonal holonomy element. This motivates the definition $r^\sharp := \min(r - \dim \mathcal{K}, 1)$.

Without loss of generality assume $r^\sharp_{2,1} \geq r^\sharp_{1,2}$ (the opposite case is symmetric). Define

$$\mathcal{V} \;:=\; \mathcal{W}^{2,1} \cap \mathcal{U},$$

where $\mathcal{W}^{2,1}$ is the non-empty open set introduced in (5.11) and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ is the dense open subset constructed above. Since $\mathcal{W}^{2,1}$ is non-empty and open and $\mathcal{U}$ is dense, $\mathcal{V}$ is a non-empty open subset of M.

Consider the mixed curvature span

$$\mathcal{S}_p \;:=\; \operatorname{span}\bigl\{R^C(X, Y) : X \in V_1,\ Y \in V_2\bigr\}.$$

By the Ambrose–Singer theorem [3], $\mathcal{S}_p \subseteq \mathfrak{hol}_p(\nabla^C)$. We now show that $\mathcal{S}_p$ contains at least one purely off-diagonal element for every $p \in \mathcal{V}$.

Suppose $r^\sharp_{2,1} = 1$ and fix $p \in \mathcal{V}$. Since $p \in \mathcal{W}^{2,1}$, we have $h(p) \neq 0$ and $\tau_p \neq 0$. When $p \notin \mathcal{W}^{1,2}$ (i.e. $T^{1,2}_p = 0$), the off-diagonal projection receives contributions from $T^{2,1}$ alone, and the argument proceeds exactly as in the single-component case below. We therefore concentrate on the more delicate situation $p \in \mathcal{W}^{2,1} \cap \mathcal{W}^{1,2}$, where both $T^{2,1}_p \neq 0$ and $T^{1,2}_p \neq 0$.

Write $\tilde h = |T^{1,2}|_p$, $\tilde\sigma = T^{1,2}_p / |T^{1,2}|_p$, and choose adapted orthonormal bases $\{e_1, e_2 = J_1 e_1\}$ of V_1 and $\{f_1, f_2 = J_2 f_1\}$ of V_2 with

$$e_1 = \frac{\tilde\sigma^\sharp}{|\tilde\sigma|}, \qquad f_1 = \frac{\tau^\sharp}{|\tau|},$$

so that $\tau = a\, f_1^*$ and $\tilde\sigma = c\, e_1^*$ with $a = |\tau| > 0$ and $c = |\tilde\sigma| > 0$. In these bases the diagonal and off-diagonal projections of $R^C(X, Y)$ for mixed inputs $X \in V_1$, $Y \in V_2$, acting on $Z \in V_1$, decompose as

$$R^C(X, Y)Z \;=\; \underbrace{\Phi(X, Y)}_{\in\, \mathfrak{g}_0}(Z) \;+\; \underbrace{\Xi(X, Y)}_{\in\, \mathfrak{g}_1}(Z), \tag{5.18}$$

where $\mathfrak{g}_0 = \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2)$ and $\mathfrak{g}_1 = \operatorname{Hom}(V_1, V_2) \oplus \operatorname{Hom}(V_2, V_1)$. The diagonal projection Φ takes the form $\Phi(X, Y) = \lambda_{\text{tot}}(X, Y)\, J_1 + \mu_{\text{tot}}(X, Y)\, J_2$, so $\ker \Phi = \ker \lambda_{\text{tot}} \cap \ker \mu_{\text{tot}}$ has dimension at least 2 in $V_1 \otimes V_2 \cong \mathbb{R}^4$. Note that the cross-terms $K^{(2,1)}_X \circ K^{(1,2)}_Y$ and $K^{(1,2)}_Y \circ K^{(2,1)}_X$ map $V_1 \to V_1$ (direct computation), so they contribute only to Φ and not to Ξ.

The off-diagonal map $\Xi(X, Y)(Z) \in V_2$ receives four types of contributions: a linear term and a quadratic term from each of $T^{2,1}$ and $T^{1,2}$. We decompose $\Xi = \Xi_{\text{quad}} + \Xi_{\text{lin}}$, where Ξ_{quad} collects the purely quadratic torsion terms (depending only on the pointwise values of T) and Ξ_{lin} collects the terms involving covariant derivatives of T. In the adapted bases with $\tau = a\, f_1^*$ and $\tilde{\sigma} = c\, e_1^*$, a direct calculation shows that the matrix of Ξ_{quad} with respect to the ordered basis $\{e_1\otimes f_1,\, e_1\otimes f_2,\, e_2\otimes f_1,\, e_2\otimes f_2\}$ of $V_1\otimes V_2$ and the identification $\mathrm{Hom}(V_1, V_2) \cong \mathbb{R}^4$ is

$$\Xi_{\mathrm{quad}} \;=\; \mathrm{diag}\Big(\frac{a^2h^2 + c^2\tilde{h}^2}{4},\; \frac{c^2\tilde{h}^2}{4},\; \frac{a^2h^2}{4},\; 0\Big). \tag{5.19}$$

In particular, Ξ_{quad} has rank exactly 3 with $\ker \Xi_{\mathrm{quad}} = \mathrm{span}\{e_2 \otimes f_2\}$. The three nonzero diagonal entries arise as follows: the $(1,1)$-entry equals $c^2\tilde{h}^2/4$ from the pure $T^{1,2}$-quadratic term $\frac{\tilde{h}^2}{4}\tilde{\sigma}(X)\tilde{\sigma}(Z)\, Y$ evaluated at $X = Z = e_1$, $Y = f_2$; the $(2,2)$-entry equals $a^2h^2/4$ from the pure $T^{2,1}$-quadratic term $\frac{h^2}{4}\tau(Y)g(X,Z)\tau^\sharp$ at $X = Z = e_2$, $Y = f_1$; and the $(0,0)$-entry is their sum. Crucially, all three entries depend only on the pointwise values h, $\tilde{h}$, $a = |\tau|$, $c = |\tilde{\sigma}|$ and receive no corrections from covariant derivatives of T. This is because the relevant inner products $g(J_1X, Z)$ and the J_2-components vanish in the corresponding matrix positions by the adapted basis construction (for instance, $g(J_1e_1, e_1) = g(e_2, e_1) = 0$ and the f_2-component of $J_2f_2 = -f_1$ is zero).

It follows that the three basis elements $e_1 \otimes f_1$, $e_1 \otimes f_2$, $e_2 \otimes f_1$ are not in $\ker \Xi_p$ at any point of $\mathcal{W}^{2,1}\cap\mathcal{W}^{1,2}$, regardless of the values of the covariant derivatives of T.

We now produce the required purely off-diagonal element. Adding derivative-dependent terms to Ξ_{quad} cannot reduce the rank to 0 (rank 0 would force all diagonal entries to vanish, contradicting the derivative-independence just established); hence $\mathrm{rank}\,\Xi_p \geq 1$ on $\mathcal{W}^{2,1} \cap \mathcal{W}^{1,2}$. If $\mathrm{rank}\,\Xi_p \geq 3$, then $\dim\ker\Xi_p \leq 1 < 2 \leq \dim\ker\Phi_p$, so $\ker\Phi_p \not\subset \ker\Xi_p$; picking any $\xi \in \ker\Phi_p \setminus \ker\Xi_p$ gives $R^C(\xi) \in \mathcal{S}_p \cap \mathfrak{so}^{\mathrm{off}}(T_pM) \setminus \{0\}$.

It remains to handle the case $\mathrm{rank}\,\Xi_p = 2$, where $\ker\Xi_p$ is two-dimensional and the inclusion $\ker\Phi_p \subset \ker\Xi_p$ is not excluded by dimension alone. This inclusion holds if and only if the two-dimensional subspaces $\ker\Phi_p$ and $\ker\Xi_p$ of $V_1 \otimes V_2 \cong \mathbb{R}^4$ coincide. A key structural feature of the product geometry prevents this from occurring on a dense open set. The off-diagonal map Ξ depends on the *transverse* derivatives of the torsion, namely $\rho_Y = \nabla^{g_2}_Y(h\tau)$ for $Y \in V_2$ and $\tilde{\rho}_X = \nabla^{g_1}_X(\tilde{h}\tilde{\sigma})$ for $X \in V_1$; while the diagonal map Φ depends on the *longitudinal* derivatives, namely $\nabla^{g_1}_X(h\tau)$ for $X \in V_1$ and $\nabla^{g_2}_Y(\tilde{h}\tilde{\sigma})$ for $Y \in V_2$. On a product $M_1 \times M_2$, derivatives along V_1 and along V_2 of the same section are algebraically independent in the first jet bundle, since they correspond to partial derivatives in orthogonal fibre directions. Thus $\ker\Xi_p$ and $\ker\Phi_p$ are determined by independent geometric data, and their coincidence imposes a non-trivial analytic constraint.

To see that this constraint is not identically satisfied, observe that the condition $\mathrm{rank}\,\Xi_p \leq 2$ is equivalent to the vanishing of all 3×3 minors of Ξ_p; since Ξ_{quad} has rank 3, these minors evaluated at zero transverse derivatives are non-zero, hence the minors are not identically zero as functions of the transverse derivatives, and their vanishing imposes non-trivial conditions. At any point where these conditions fail—that is, where $\operatorname{rank}\Xi_p = 3$—the dimensional argument above applies directly. Such points exist: on flat tori, harmonic 1-forms are parallel by the Bochner theorem, so if the harmonic part of $h\tau$ dominates, the transverse derivatives $\nabla^{g_2}(h\tau)$ are small and the rank is 3; on non-flat factors ($\kappa_i \neq 0$), a direct argument using the Gauss curvature contributions to $\mathfrak{hol}_p \cap \mathfrak{g}_0$ yields $\mathfrak{hol}_p \supset \mathfrak{g}_0$, and any element of $\mathcal{S}_p$ with $\pi_1 \neq 0$ then produces a purely off-diagonal element by subtraction. In all cases, the set $\{p \in \mathcal{W}^{2,1} \cap \mathcal{U} : \operatorname{rank}\Xi_p \geq 3\}$ is open (the rank of a smooth linear map is lower semicontinuous) and dense. Density is established by the same Greene–Jacobowitz approximation scheme used in Lemma 5.1: for a real-analytic metric, the locus $\{\operatorname{rank}\Xi \leq 2\}$ is a proper analytic subset (since $\operatorname{rank}\Xi_{\text{quad}} = 3$ furnishes a point in the jet space where all relevant minors are nonzero, and the identity principle for analytic functions applies); for a smooth metric, C^2-approximation by analytic metrics transfers the density to the limit. On this dense open set, $\dim\ker\Xi_p \leq 1 < 2 \leq \dim\ker\Phi_p$, so $\ker\Phi_p \not\subset \ker\Xi_p$.

In either case, we obtain a nonzero $\xi \in \ker\Phi_p \setminus \ker\Xi_p$ and hence

$$R^C(\xi) \;\in\; \mathcal{S}_p \cap \mathfrak{so}^{\text{off}}(T_pM) \setminus \{0\}. \tag{5.20}$$

When $p \notin \mathcal{W}^{1,2}$ (so $T_p^{1,2} = 0$), the diagonal projection reduces to $\lambda(X,Y)\,J_1$ and the off-diagonal map receives contributions only from $T^{2,1}$. The original argument applies: pick $Y \in V_2$ with $\tau(Y) \neq 0$ and choose $X_0 \in \ker(X \mapsto \lambda(X,Y))$; then $R^C(X_0,Y)$ is purely off-diagonal and nonzero by the orthogonality of $g(J_1X_0,\cdot)$ and $g(X_0,\cdot)$. The symmetric argument, exchanging the roles of V_1 and V_2, applies when $r^\sharp_{1,2} = 1$ and produces a nonzero purely off-diagonal element in $\operatorname{Hom}(V_2,V_1)$. Each construction independently yields at least one purely off-diagonal element on its respective open set.

Since $\mathcal{S}_p \cap \mathfrak{so}^{\text{off}}(T_pM) \subseteq \mathfrak{hol}_p(\nabla^C) \cap \mathfrak{so}^{\text{off}}(T_pM) = \mathfrak{hol}_p^{\text{off}}(\nabla^C)$ (Definition 3.2), we conclude that for every $p \in \mathcal{V}$,

$$\dim\mathfrak{hol}_p^{\text{off}}(\nabla^C) \;\geq\; r^\sharp \;=\; \min\bigl(\operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) - \dim\mathcal{K},\ 1\bigr). \tag{5.21}$$

In particular, if $r^\sharp > 0$, the holonomy representation at every point of the non-empty open set $\mathcal{V}$ is irreducible with respect to $V_1 \oplus V_2$, therefore, we have global irreducibility. The lower bound depends only on $[\omega]_{\text{mixed}}$ and on the parallel-form strata $\mathcal{P}^k(M_i)$; it is therefore invariant under any product-metric deformation that preserves these strata. □

Remark 5.3 (Necessity of the Non-Parallel Torsion Condition)**.** Theorem 5.2 yields a non-trivial lower bound only when the linear term

$$\tfrac{1}{2}(\nabla^{\text{LC}}T)_{\text{off}}$$

is non-zero. If instead $\nabla^{\text{LC}}T = 0$, the curvature reduces to

$$R^C = R^{\text{LC}} + \tfrac{1}{4}Q_T,$$

and there is no guarantee that the purely quadratic off-diagonal part $(Q_T)_{\mathrm{off}}$ generates $r^\sharp$ independent directions.

Consider for instance the flat torus $M = T^2 \times S^1$ with the constant volume form $T = dx \wedge dy \wedge dz$, which is harmonic and satisfies

$$\mathrm{rank}_{\mathbb{R}}([T]_{\mathrm{mixed}}) = 1.$$

Note that the following example lies outside the *admissible* class but illustrates why the non-parallel torsion condition is essential.

Since T is parallel, $\nabla^{\mathrm{LC}} T = 0$; hence every simple tensor in any minimal harmonic decomposition is completely parallel. In the notation of Theorem 5.2 we have

$$\mathcal{V}_{2,1} = \mathrm{span}\{dx \wedge dy \otimes dz\}, \qquad \mathcal{P}_2(T^2) = \mathrm{span}\{dx \wedge dy\}, \qquad \mathcal{P}_1(S^1) = \mathrm{span}\{dz\},$$

so that

$$\ker \Psi_p = \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(T^2) \otimes \mathcal{P}_1(S^1)\big) = \mathcal{V}_{2,1}, \qquad \ker \widetilde{\Psi}_p = 0.$$

Thus $\dim \mathcal{K} = 1$ and consequently

$$r^\sharp = \mathrm{rank}_{\mathbb{R}}([T]_{\mathrm{mixed}}) - \dim \mathcal{K} = 1 - 1 = 0.$$

A direct computation gives $R^C = 0$, hence

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) = 0, \qquad r^\sharp = 0.$$

The bound $0 \geq 0$ is saturated but trivial, illustrating that without the non-parallel torsion condition the theorem cannot force irreducibility.

Remark 5.4 (Geometric Significance of Invariance)**.** The invariance of the lower bound $r^\sharp$ is a non-trivial property that holds under the Assumption 2.2. In those cases, the bound depends only on:

- the mixed class $[\omega]_{\mathrm{mixed}}$,
- the parallel-form strata $\mathcal{P}_k(M_i)$ of the factor metrics.

Consequently, $r^\sharp$ is invariant under any deformation of g_1 or g_2 that preserves these strata. This stability is relevant for manifolds with special holonomy where $\mathcal{P}_k(M_i)$ are non-trivial.

Remark 5.5 (Forced irreducibility and quantified off-diagonal holonomy)**.** Since ∇^C is metric, any non-trivial off-diagonal holonomy implies that $\mathfrak{hol}_p(\nabla^C)$ acts irreducibly on T_pM. Theorem 5.2 quantifies this "forced irreducibility" principle in terms of the mixed tensor rank, modulo the obstruction space

$$\mathcal{K} = \ker \Psi_p \oplus \ker \widetilde{\Psi}_p,$$

where, in the notation of the theorem,

$$\ker \Psi_p = \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big), \qquad \ker \widetilde{\Psi}_p = \mathcal{V}_{1,2} \cap \big(\mathcal{P}_1(M_1) \otimes \mathcal{P}_2(M_2)\big).$$

Specifically, the dimension of the off-diagonal holonomy subspace satisfies

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \geq r^\sharp = \min\big(\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K},\ 1\big),$$

and irreducibility is forced precisely when $r^\sharp > 0$, i.e. when

$$\operatorname{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) > \dim \mathcal{K}.$$

In other words, a non-zero value of the corrected rank $r^\sharp$ guarantees that the holonomy representation cannot split according to the product decomposition $T_pM = V_1 \oplus V_2$; the connection is geometrically "entangled" across the two factors.

Remark 5.6 (Analytic approximation as a tool of proof)**.** The proofs of Lemma 5.1 and Theorem 5.2 make essential use of real-analytic approximations of the smooth metric. This technique does not restrict the generality of the results, because real-analytic metrics are dense in the C^∞ topology (Greene–Jacobowitz [5]) and the relevant quantities – the dimension of the space of parallel forms, the harmonic representatives of fixed cohomology classes, and the covariant derivatives of those harmonic forms – depend continuously on the metric in the appropriate topologies.

In Lemma 5.1 the reduction to an analytic metric allows us to apply the identity principle for analytic functions: if an analytic section of a vector bundle vanishes on an open set, it vanishes identically. This principle is used to show that a relation among the covariant derivatives of harmonic forms on an open set forces the existence of a parallel combination of the forms, contradicting the hypothesis of linear independence modulo parallel forms. For a smooth metric, we approximate by analytic metrics, prove the existence of a dense open set of maximal rank for each approximation, and then pass to the limit using C^1 convergence of the covariant derivatives.

In Theorem 5.2 the analytic approximation is used only indirectly, through Lemma 5.1. Once the lemma guarantees a dense open set where the linear maps Ψ_p and $\widetilde{\Psi}_p$ have maximal rank, the inequality $\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \geq r^\sharp$ follows on that open set.

Thus analyticity serves solely as a convenient tool to handle the delicate rank conditions in Lemma 5.1; the final statements are valid for arbitrary smooth product metrics and do not require any analyticity assumption on the metric or the torsion.

Remark 5.7 (Dichotomy of the obstruction)**.** For admissible product manifolds, the obstruction space $\mathcal{K}$ is controlled by a single topological datum: the vanishing or non-vanishing of $\mathcal{P}_1(M_i)$. Since $\mathcal{P}_2(M_i) = \mathbb{R} \cdot \mathrm{vol}_{M_i}$ for every compact oriented surface, the obstruction kernels reduce to

$$\ker \Psi_p \neq \{0\} \iff \beta \in \mathcal{P}_1(M_2), \qquad \ker \widetilde{\Psi}_p \neq \{0\} \iff \alpha \in \mathcal{P}_1(M_1),$$

where α, β are the harmonic forms appearing in the minimal decomposition of $[\omega]_{\mathrm{mixed}}$. A compact oriented surface F satisfies $\mathcal{P}_1(F) = \{0\}$ if and only if either $g(F) \geq 2$ (by the Poincaré–Hopf theorem: $\chi(F) \neq 0$ forbids nowhere-vanishing vector fields, hence parallel 1-forms) or $g(F) = 1$ and the metric is not flat (a non-zero parallel 1-form on a surface forces the Gaussian curvature to vanish identically).

This yields a sharp dichotomy. When at least one factor has genus $g \geq 2$, the obstruction space $\mathcal{K}$ is at most one-dimensional regardless of the metric, and $r^\sharp = 1$ whenever the mixed class $[\omega]_{\text{mixed}}$ is non-trivial in the corresponding Künneth component; in particular, on $\Sigma_{g_1} \times \Sigma_{g_2}$ with $g_1, g_2 \geq 2$, the bound $r^\sharp = 1$ is *purely topological* and forces irreducibility for every cohomologically calibrated connection with non-parallel torsion representing a non-trivial mixed class, independently of the choice of product metric. The only setting where the metric controls the transition $r^\sharp = 0 \leftrightarrow r^\sharp = 1$ is when the relevant factor is a torus T^2: the parallel-form stratum $\mathcal{P}_1(T^2, g)$ jumps from $\mathbb{R}^2$ to $\{0\}$ as soon as the Gaussian curvature becomes non-identically-zero, and this jump is the sole mechanism by which the bound changes.

Finally, we note that the untruncated quantity $r - \dim \mathcal{K}$ counts the number of independent mixed directions that are not absorbed by parallel forms. In the admissible setting this integer can reach the value 2 (when both Künneth components contribute and $\mathcal{K} = \{0\}$), but Theorem 5.2 transfers only $\min(r - \dim \mathcal{K}, 1)$ of these directions into $\mathfrak{hol}^{\text{off}}$ as purely off-diagonal elements. The truncation arises because the current proof constructs at most one purely off-diagonal holonomy element per bidegree component at a single point; the question of whether this truncation can be removed is discussed in Section 7.

6. Illustrative Examples

The case $\Sigma_g \times T^2$ (Example of a non-trivial lower bound). Fix $g \geq 2$ and endow Σ_g with any hyperbolic metric and T^2 with the flat metric. Harmonic spaces:

$$H^1(T^2) = \text{span}\{dx, dy\}, \qquad H^2(\Sigma_g) = \text{span}\{\text{vol}_\Sigma\},$$

and choose a non-zero harmonic 1-form $\alpha \in H^1(\Sigma_g)$ (such an α exists because $b_1(\Sigma_g) = 2g > 0$).
For the hyperbolic metric on Σ_g we have

$$\mathcal{P}_1(\Sigma_g) = 0, \qquad \mathcal{P}_2(\Sigma_g) = \text{span}\{\text{vol}_\Sigma\} \cong \mathbb{R},$$

since the volume form is harmonic and parallel.
For the flat metric on T^2 all harmonic forms are parallel, hence

$$\mathcal{P}_1(T^2) = H^1(T^2) \cong \mathbb{R}^2, \qquad \mathcal{P}_2(T^2) = H^2(T^2) \cong \mathbb{R}.$$

Consider the mixed cohomology class

$$[\omega] = \underbrace{\text{vol}_\Sigma \otimes dx}_{\in H^2(\Sigma_g) \otimes H^1(T^2)} + \underbrace{\alpha \otimes \text{vol}_{T^2}}_{\in H^1(\Sigma_g) \otimes H^2(T^2)} \in H^3(\Sigma_g \times T^2).$$

Both Künneth components are already simple tensors, therefore their minimal tensor ranks are

$$r_{2,1} = 1, \qquad r_{1,2} = 1,$$

and the mixed rank equals

$$\text{rank}_\mathbb{R}\big([\omega]_{\text{mixed}}\big) = r_{2,1} + r_{1,2} = 2.$$

The obstruction kernels are computed as follows. For the $(2,1)$-part $\pi_{2,1}([\omega]) = \mathrm{vol}_\Sigma \otimes dx$,

$$\ker \Psi_p = \mathcal{V}_{2,1} \cap \big(\mathcal{P}_2(M_1) \otimes \mathcal{P}_1(M_2)\big) = \mathrm{span}\{\mathrm{vol}_\Sigma \otimes dx\},$$

because $\mathrm{vol}_\Sigma \in \mathcal{P}_2(\Sigma_g)$ and $dx \in \mathcal{P}_1(T^2)$. Hence $\dim \ker \Psi_p = 1$.
For the $(1,2)$-part $\pi_{1,2}([\omega]) = \alpha \otimes \mathrm{vol}_{T^2}$,

$$\ker \widetilde{\Psi}_p = \mathcal{V}_{1,2} \cap \big(\mathcal{P}_1(M_1) \otimes \mathcal{P}_2(M_2)\big) = \{0\},$$

because $\mathcal{P}_1(M_1) = \{0\}$ and $\mathcal{P}_2(M_2) = \mathrm{span}\{\mathrm{vol}_{T^2}\}$, hence $\mathcal{P}_1(M_1)\otimes\mathcal{P}_2(M_2) = \{0\}$. Hence $\dim \ker \widetilde{\Psi}_p = 0$. Thus

$$\mathcal{K} = \ker \Psi_p \oplus \ker \widetilde{\Psi}_p, \qquad \dim \mathcal{K} = 1 + 0 = 1,$$

and the corrected rank is

$$r^\sharp = \min\big(\mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K},\ 1\big) = \min(2-1,\ 1) = 1.$$

Theorem 5.2 therefore yields the non-trivial lower bound

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \ \geq\ 1.$$

In particular, any cohomologically calibrated metric connection with non-parallel torsion representing $[\omega]$ necessarily has irreducible holonomy across the product splitting $\Sigma_g \times T^2$.

The case $T^2 \times T^2$ (Example of a trivial lower bound). Let $M = T^2 \times T^2$ endowed with the flat product metric $g = g_1 \oplus g_2$. Fix harmonic bases

$$H^1(T^2) = \mathrm{span}\{dx, dy\}, \quad H^2(T^2) = \mathrm{span}\{vol\}.$$

For the flat metric we have

$$\mathcal{P}_1(T^2) = H^1(T^2) \cong \mathbb{R}^2, \qquad \mathcal{P}_2(T^2) = H^2(T^2) \cong \mathbb{R}.$$

The mixed Künneth components are

$$H^2(T^2) \otimes H^1(T^2) \cong \mathbb{R}^2, \quad H^1(T^2) \otimes H^2(T^2) \cong \mathbb{R}^2.$$

Consider the single cohomology class

$$[\omega] = \underbrace{vol \otimes dx}_{(2,1)} + \underbrace{dy \otimes vol}_{(1,2)}.$$

Its minimal tensor-rank decomposition is

$$\pi_{2,1}([\omega]) = vol \otimes dx \quad (\text{rank } 1), \qquad \pi_{1,2}([\omega]) = dy \otimes vol \quad (\text{rank } 1).$$

Hence

$$\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1 + 1 = 2.$$

Since $vol \in \mathcal{P}_2$ and $dx, dy \in \mathcal{P}_1$, both simple tensors lie in the obstruction spaces

$$\ker \Psi_p = \mathrm{span}\{vol \otimes dx\}, \qquad \ker \tilde{\Psi}_p = \mathrm{span}\{dy \otimes vol\},$$

so $\dim \mathcal{K} = 2$. Theorem 5.2 gives

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \geq \min(2-2,\ 1) = 0.$$

7. Conclusion

We have shown that for *admissible* product manifolds $M = M_1 \times M_2$, specifically, products of compact oriented surfaces, and for any cohomologically calibrated metric connection ∇^C with non-parallel torsion, the off-diagonal holonomy dimension admits a quantified lower bound on a non-empty open subset $\mathcal{V} \subset M$

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \;\geq\; r^\sharp = \min(r - \dim \mathcal{K},\, 1),$$

where $r = \operatorname{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}})$ and $\mathcal{K} = \ker \Psi_p \oplus \ker \widetilde{\Psi}_p$ is the sum of the kernels of the linear obstruction maps. The bound is metric-invariant under deformations preserving the parallel-form strata $\mathcal{P}_k(M_i)$. This provides the first *computable* obstruction to reducible holonomy on product manifolds within the admissible class. As shown in Remark 5.7, the behaviour of $r^\sharp$ across the admissible class is completely determined: $r^\sharp = 1$ is forced by topology alone whenever at least one factor has genus $g \geq 2$ and the mixed class is non-trivial, while the transition $r^\sharp = 0 \leftrightarrow 1$ occurs only when a torus factor changes from flat to non-flat.

The examples in Section 6 illustrate both non-trivial and trivial cases, while Remark 2.4 shows by explicit construction that the bound fails to be well-defined without the admissibility hypothesis, establishing sharpness of our assumptions.

When the torsion is purely harmonic and the parallel-form strata are minimal, the bound is often saturated (e.g., on flat tori with parallel torsion). It is natural to conjecture that equality $\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) = r^\sharp$ holds generically in this setting, while strict inequality may occur when the co-exact/exact parts of T generate additional off-diagonal curvature or when the parallel-form spaces are larger. A related but independent question concerns the truncation $\min(\cdot, 1)$ in the definition of $r^\sharp$: the untruncated quantity $r - \dim \mathcal{K}$ is a well-defined topological invariant that can exceed 1 (already in the admissible setting it can reach 2, and beyond the admissible class it can be arbitrarily large). We conjecture that the stronger bound $\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^C) \geq r - \dim \mathcal{K}$ holds without the truncation; establishing this would require new techniques for producing multiple linearly independent purely off-diagonal holonomy elements at a single point. A systematic investigation of both the sharpness and the potential strengthening of the bound remains an interesting open problem.

A further direction is to explore extensions beyond the admissible setting. Imposing *Kruskal-rank constraints* on harmonic representatives could yield intrinsic mixed factor spaces even when $b_2(M_i) > 1$, potentially treating Calabi–Yau threefolds or $G_2 \times S^1$ backgrounds. Such extensions would require new algebraic tools and are left for future investigation.

Another further interesting direction for future work could be to explore the implications of these results in the context of deformations of complex structures, a field pioneered by the works of Kodaira [7] and Kodaira-Spencer [8].

Compactness and Hodge theory. Compactness ensures the existence and uniqueness of harmonic representatives [10], and the vanishing of boundary terms in integrations by parts. Extending the results to noncompact manifolds would require an L^2 Hodge framework or alternative analytic hypotheses.

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