A family of special case of sequential warped-product manifolds with semi-Riemannian Einstein metrics

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Abstract

We derive the general formulas for a special configuration of the sequential warped-product semi-Riemannian manifold to be Einstein, where the base-manifold is the product of two manifolds both equipped with a generic diagonal conformal metrics. Subsequently we study the case in which these two manifolds are conformal to a n_1-dimensional and n_2-dimensional pseudo-Euclidean space, respectively. For the latter case, we prove the existence of a family of solutions that are invariant under the action of a $(n_1 - 1)$-dimensional group of transformations to the case of positive constant Ricci curvature ($\lambda > 0$).

1. Introduction and Preliminaries

The warped-product manifolds are type of manifolds introduced by Bishop and O'Neill [1]. These manifolds have become very important in the context of differential geometry and are also extensively studied in the arena of General Relativity, for instance with respect to generalized Friedmann-Robrtson-Walker spacetimes. Many properties for warped product manifolds and submanifolds were presented by B.-Y. Chen in [2].

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A warped-product manifold can be constructed as follows. Let (B, g_B) and (F, g_F) be two semi-Riemannian manifolds and τ, σ be the projection of $B \times F$ onto B and F, respectively.
The warped-product $M = B \times_f F$ is the manifold $B \times F$ equipped with the metric tensor $g = \tau^* g_B + f^2 \sigma^* g_F$, where * denotes the pullback and f is a positive smooth function on B, the so-called warping function.

Explicitly, if X is tangent to $B \times F$ at (p, q) (where p is a point on B and q is a point on F), then:

$$\langle X, X\rangle = \langle d\tau(X), d\tau(X)\rangle + f^2(p)(d\sigma(X), d\sigma(X)).$$

B is called the *base-manifold* of $M = B \times_f F$ and F is the *fiber-manifold*. If $f = 1$, then $B \times_f F$ reduces to a semi-Riemannian product manifold. The leaves $B \times q = \sigma^{-1}(q)$ and the fibers $p \times F = \tau^{-1}(p)$ are Riemannian submanifolds of M. Vectors tangent to leaves are called horizontal and those tangent to fibers are called vertical. *By* $\mathcal{H}$ we denote the orthogonal projection of $T_{(p,q)}M$ onto its horizontal subspace $T_{(p,q)}(B \times q)$ and $\mathcal{V}$ denotes the projection onto the vertical subspace $T_{(p,q)}(p \times F)$, see [3].

If M is an n-dimensional manifold, and g_M is its metric tensor, the Einstein condition means that $Ric_M = \lambda g_M$ for some constant λ, where Ric_M denotes the Ricci tensor of g_M. An Einstein manifold with $\lambda = 0$ is called Ricci-flat manifolds.
Then keeping this in mind, we get that a warped-product manifold $(M, g_M) = (B, g_B) \times_f (F, g_F)$ (where (B, g_B) is the base-manifold, (F, g_F) is the fiber-manifold), with $g_M = g_B + f^2 g_F$, is Einstein if only if (see [2]):

(1.1) $$Ric_M = \lambda g_M \iff \begin{cases} Ric_B - \frac{d}{f} Hess(f) = \lambda g_B \\ Ric_F = \mu g_F \\ f\Delta f + (d-1)|\nabla f|^2 + \lambda f^2 = \mu \end{cases}$$

where λ and μ are constants, d is the dimension of F, $Hess(f)$, Δf and ∇f are, respectively, the Hessian, the Laplacian (given by $tr\ Hess(f)$) and the gradient of f for g_B, with $f : (B) \to \mathbb{R}^+$ a smooth positive function.

Contracting first equation of (1) we get:

(1.2) $$R_B f^2 - f\Delta f d = n f^2 \lambda$$

where n and R_B is the dimension and the scalar curvature of B respectively. From third equation, considering $d \neq 0$ and $d \neq 1$, we have:

(1.3) $f\Delta f d + d(d-1)|\nabla f|^2 + \lambda f^2 d = \mu d$

Now from (1.2) and (1.3) we obtain:

(1.4) $|\nabla f|^2 + [\frac{\lambda(d-n)+R_B}{d(d-1)}]f^2 = \frac{\mu}{(d-1)}$.

In 2017 de Sousa and Pina [4], studied warped-product semi-Riemannian Einstein manifolds in case that base-manifold is conformal to an n-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional group with Ricci-flat fiber F. In [5] the authors extend the work done for multiply warped space.

In [6], the author introduced a new type of warped-products called sequential warped-products, i.e. (M, g_M) where $M = (B_1 \times_h B_2) \times_f F$ and $g_M = (g_{B_1} + h^2 g_{B_2}) + f^2 g_F$, to cover a wider variety of exact solutions to Einsteins field equation.
Regarding the sequential warped-product manifolds, some works have been published in recent years ([7], [8], [9], [10], [11], [12]).

The main aim of the present paper is largely to continue to extend the work done in [4] (as was done for the multiply warped-product manifold in [5]), also for a special case of sequential warped-product manifolds, (i.e. for $h = 1$, with B_2 as an Einstein manifold, and flat fiber F, where the base-manifold $B = B_1 \times B_2$ is the product of two manifolds both equipped with a conformal metrics, and the warping function is a smooth positive function $f(x, y) = f_1(x) + f_2(y)$ where each is a function on its individual manifold). The method will be as follows: first deriving the general formulas to be Einstein and second, providing the existence of solutions that are invariant under the action of a $(n_1 - 1)$-dimensional group of transformations to the case of positive constant Ricci curvature. In fact, since in both references, [4] and [5], the authors show solutions for the Ricci-flat case ($\lambda = 0$), we, following their same construction, show the existence of a family solutions for constant positive Ricci curvature ($\lambda > 0$). In particular, this proof of the existence of a family of solutions also holds for [4] considering $dimF = dimB$.

Definition 1.1: We consider the special case of the Einstein sequential warped-product manifold, that satisfies (1.1). The manifold (M, g_M) comprises the base-manifold (B, g_B) which is a Riemannian (or pseudo-Riemannian) product-manifold $B = B_1 \times B_2$, with B_2 as an Einstein manifold (i.e., $Ric_{B_2} = \lambda g_{B_2}$, where λ is the same for (1.1) and g_{B_2}

is the metric for B_2), and $dim(B_2) = n_2$, $dim(B_1) = n_1$ the dimension of B_2 and B_1, respectively, so that $dim(B) = n = n_1 + n_2$. The warping function $f : B \to \mathbb{R}^+$ is a smooth positive function $f(x, y) = f_1(x) + f_2(y)$ (where each is a function on its individual manifold, i.e., $f_1 : B_1 \to \mathbb{R}^+$ and $f_2 : B_2 \to \mathbb{R}^+$). The fiber-manifold (F, g_F) is the $\mathbb{R}^d$, with orthogonal Cartesian coordinates such that $g_{ab} = -\delta_{ab}$.

Proposition 1.1: If we write the B-product as $B = B_1 \times B_2$, where:
i) Ric_{B_i} is the Ricci tensor of B_i referred to g_{B_i}, where $i = 1, 2$,
ii) $f(x, y) = f_1(x) + f_2(y)$, is the smooth warping function, where $f_i : B_i \to \mathbb{R}^+$,
iii) $Hess(f) = \sum_i \tau_i^* Hess_i(f_i)$ is the Hessian referred on its individual metric, where τ_i^* are the respective pullbacks, (and $\tau_2^* Hess_2(f_2) = 0$ since B_2 is Einstein),
iv) ∇f is the gradient (then $|\nabla f|^2 = \sum_i |\nabla_i f_i|^2$), and
v) $\Delta f = \sum_i \Delta_i f_i$ is the Laplacian, (from (iii) therefore also $\Delta_2 f_2 = 0$).
Then the Ricci curvature tensor will be:

$$(1.5)\quad \begin{cases} Ric_M(X_i, X_j) = Ric_{B_1}(X_i, X_j) - \frac{d}{f} Hess_1(f_1)(X_i, X_j) \\ Ric_M(Y_i, Y_j) = Ric_{B_2}(Y_i, Y_j) \\ Ric_M(U_i, U_j) = Ric_F(U_i, U_j) - g_F(U_i, U_j) f^* \\ Ric_M(X_i, Y_j) = 0 \\ Ric_M(X_i, U_j) = 0, \\ Ric_M(Y_i, U_j) = 0, \end{cases}$$

where $f^* = \frac{\Delta_1 f_1}{f} + (d-1)\frac{|\nabla f|^2}{f^2}$, and X_i, X_j, Y_i, Y_j, U_i, U_j are vector fields on B_1, B_2 and F, respectively.

Theorem 1.1: *A warped-product manifold is a special case of an Einstein sequential warped-product manifold, as defined in Definition 1.1, if and only if:*

$$(1.6)\ Ric_M = \lambda g_M \iff \begin{cases} Ric_{B_1} - \frac{d}{f}\tau_1^* Hess_1(f_1) = \lambda g_{B_1} \\ \tau_2^* Hess_2(f_2) = 0 \\ Ric_{B_2} = \lambda g_{B_2} \\ Ric_F = 0 \\ f\Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0, \end{cases}$$

(since Ric_B is the Ricci curvature of B referred to g_B, then $Ric_B = Ric_{B_1} + Ric_{B_2} = \lambda(g_{B_1} + g_{B_2}) + \frac{d}{f}\tau_1^* Hess_1(f_1)$.

Therefore from (1.2) and (1.3):

$$(1.7)\ R_M = \lambda(n+d) \Longleftrightarrow \begin{cases} R_{B_1} f - \Delta_1 f_1 d = n_1 f \lambda \\ \Delta_2 f_2 = 0 \\ R_{B_2} = \lambda n_2 \\ R_F = 0 \\ f \Delta_1 f_1 + (d-1)|\nabla f|^2 + \lambda f^2 = 0. \end{cases}$$

where n_1 and R_1 are the dimension and the scalar curvature of B_1 referred to g_{B_1}, respectively.

Proof. We applied the condition that the warped-product manifold of system (1.5) is Einstein. □

This particular type of Einstein sequential warped-product manifold, as per *Definition 1.1*, allows to cover a wider variety of exact solutions of Einstein's field equation, without complicating the calculations much, compared to the Einstein warped-product manifolds with Ricci-flat fiber (F, g_F), also considered by the authors of [4].

2. Conformal B-metrics

In this section we will consider a special type of sequential warped-product manifold (M, g_M), as described in the previous section, but in which the base-manifold is the product of two manifolds both equipped with a conformal metrics. First we will show the general formulas for which such a manifold M is Einstein, then we will show the same in the case where the conformal metrics are both diagonal, and finally for the case in which the base-manifold is the product of two conformal manifolds to a n_1-dimensional and n_2-dimensional pseudo-Euclidean space, respectively.

Theorem 2.1: *Let (B, g_B), be the base-manifold $B = (B_1 \times B_2)$, $B_1 = \mathbb{R}^{n_1}$, with coordinates $(x_1, x_2, ..x_{n_1})$, $B_2 = \mathbb{R}^{n_2}$, with coordinates $(y_1, y_2, ..y_{n_2})$, where $n_1, n_2 \geq 3$, and let $g_B = g_{B_1} + g_{B_2}$ be the metrics on B, where $g_{B_1} = \varepsilon_i \delta_{ij}$ and $g_{B_2} = \varepsilon_l \delta_{lr}$. Let $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, $f_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, $\varphi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, be smooth functions, where f_1 and f_2 are positive functions, such that $f = f_1 + f_2$ as in Definition 1.1. Finally, let (M, g_M) be $((B_1 \times B_2) \times_{f=f_1+f_2} F, g_M)$, with $g_M = \bar{g}_B + (f_1 + f_2)^2 g_F$, with conformal metric $\bar{g}_B = \bar{g}_{B_1} + \bar{g}_{B_2}$, where $\bar{g}_{B_1} = \frac{1}{\varphi_1^2} g_{B_1}$, $\bar{g}_{B_2} = \frac{1}{\varphi_2{}^2} g_{B_2}$, and $F = \mathbb{R}^d$ with $g_F = -\delta_{ab}$. Then the warped-product metric $g_M = \bar{g}_B + (f_1 + f_2)^2 g_F$ is Einstein with constant Ricci* *curvature λ if and only if, the functions f_1, f_2, φ_1 and φ_2 satisfy:*

(I) $(n_1-2)f\varphi_{1,x_ix_j} - \varphi_1 f_{1,x_ix_j}d - \varphi_{1,x_i}f_{1,x_j}d - \varphi_{1,x_j}f_{1,x_i}d = 0$ *for* $i \neq j$,

(II) $(n_2-2)\varphi_{2,y_ly_r} = 0$ *for* $l \neq r$,

(III) $\varphi_1[(n_1-2)f\varphi_{1,x_ix_i} - \varphi_1 f_{1,x_ix_i}d - 2\varphi_{1,x_i}f_{1,x_i}d]+$

$$+\varepsilon_i[f\varphi_1 \sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_kx_k} - (n_1-1)f\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_k}^2 + \varphi_1 d\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_k}f_{1,x_k}] = \varepsilon_i\lambda f,$$

(IV) $\varphi_2(n_2-2)\varphi_{2,y_ly_l} + \varepsilon_l\varphi_2\sum_{s=1}^{n^2}\varepsilon_s\varphi_{2,y_sy_s} - (n_2-1)\varepsilon_l\sum_{s=1}^{n_2}\varepsilon_s\varphi_{2,y_s}^2 = \lambda\varepsilon_l$,

(V) $-f{\varphi_1}^2\sum_{k=1}^{n_1}\varepsilon_k f_{1,x_kx_k} + (n_1-2)f\varphi_1\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_k}f_{1,x_k}+$

$$-(d-1)({\varphi_1}^2\sum_{k=1}^{n_1}\varepsilon_k f_{1,x_k}^2 + {\varphi_2}^2\sum_{s=1}^{n_2}\varepsilon_s f_{2,y_s}^2) = \lambda f^2.$$

Before proving *Theorem 2.1*, and showing the existence of a solution for $\lambda > 0$, we want to deduce the formulas for generic diagonal conformal metrics g_{B_1} and g_{B_2}. Based on this, we consider (B, g_B), the base-manifold $B = (B_1 \times B_2)$, with $dim(B_1) = n_1$, $dim(B_2) = n_2$, and $g_B = g_{B_1} + g_{B_2}$. We also consider $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, $f_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, $\varphi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, are smooth functions, where f_1 and f_2 are positive functions, such that $f = f_1 + f_2$ as in *Definition 1.1.* And finally, we consider (M, g_M) with $((B_1 \times B_2) \times_{(f_1+f_2)} F, g_M)$, with $g_M = \bar{g}_B + (f_1+f_2)^2 g_F$, with conformal metric $\bar{g}_B = \bar{g}_{B_1} + \bar{g}_{B_2}$, where $\bar{g}_{B_1} = \frac{1}{\varphi_1^2}g_{B_1}$, $\bar{g}_{B_2} = \frac{1}{{\varphi_2}^2}g_{B_2}$, and $F = \mathbb{R}^d$ with $g_F = -\delta_{ab}$.

From (1.6), considering the conformal metric on B_1 and B_2, it is easy to deduce that M is Einstein if and only if:
(2.1) $Ric_{\bar{B}_1} = \lambda\bar{g}_{B_1} + \frac{d}{f}Hess_{\bar{1}}(f_1)$, or equivalently (2.2) $R_{\bar{B}_1} = \lambda n_1 + \frac{d}{f}\Delta_{\bar{1}}(f_1)$,
(2.3) $Ric_{\bar{B}_2} = \lambda\bar{g}_{B_2}$, or equivalently (2.4) $R_{\bar{B}_2} = \lambda n_2$,
(2.5) $0 = \lambda f^2 + f\Delta_{\bar{1}}f_1 + (d-1)[|\nabla_{\bar{1}}f_1|^2 + |\nabla_{\bar{2}}f_2|^2]$.

If we consider a generic diagonal metric, $\bar{g}_{B_{ij}} = \bar{g}_{B_{1_{ij}}} + \bar{g}_{B_{2ij}} = \eta_{ij}$, and $\eta_{ij} = 0$ for $i \neq j$, then M is Einstein if and only if (2.1), (2.3) (or equivalently (2.2), (2.4)), (2.5) and the following, are satisfied:

(2.6) $Ric_{\bar{B}_1} = \frac{d}{f} Hess_{\bar{1}}(f_1)$, for $i \neq j$,

(2.7) $Ric_{\bar{B}_2} = 0$, for $i \neq j$.

Proof of Theorem 2.1. At this point we can calculate:

(2.8) $Ric_{\bar{B}_1} = \frac{1}{\varphi_1^2}\{(n_1-2)\varphi_1 Hess_1(\varphi_1) + [\varphi_1 \Delta_1 \varphi_1 - (n_1-1)|\nabla_1\varphi_1|^2] g_{B_1}\}$,

(2.9) $Ric_{\bar{B}_2} = \frac{1}{\varphi_2^2}\{(n_2-2)\varphi_2 Hess_2(\varphi_2) + [\varphi_2 \Delta_2 \varphi_2 - (n_2-1)|\nabla_2\varphi_2^2] g_{B_2}\}$,

so we can write:

(2.10) $Ric_{\bar{B}_1}(X_i, X_j) = \frac{1}{\varphi_1^2}\{(n_1-2)\varphi_1 Hess_1(\varphi_1)(X_i,X_j) + [\varphi_1\Delta_1\varphi_1 - (n_1-1)|\nabla_1\varphi_1|^2] g_{B_1}(X_i,X_j)\}$,

(2.11) $Ric_{\bar{B}_2}(Y_l, Y_r) = \frac{1}{\varphi_2^2}\{(n_2-2)\varphi_2 Hess_2(\varphi_2)(Y_l,Y_r) + [\varphi_2\Delta_2\varphi_2 - (n_2-1)|\nabla_2\varphi_2|^2] g_{B_2}(Y_l,Y_r)\}$,

(2.12) $Ric_M(X_i,X_j) = Ric_{\bar{B}_1}(X_i,X_j) - \frac{d}{f} Hess_{\bar{1}}(f_1)(X_i,X_j)$,

for what was stated in *Proposition 1.1* we have:

(2.13) $Ric_M(Y_l,Y_r) = Ric_{\bar{B}_2}(Y_l,Y_r)$,

and in the end

(2.14) $Ric_M(X_i,Y_j) = 0$.

(2.15) $Ric_M(X_i,U_j) = 0$.

(2.16) $Ric_M(Y_i,U_j) = 0$.

Since $Ric_F = 0$ we obtain:

(2.17) $Ric_M(U_i,U_j) = -g_M(U_i,U_j)(\frac{\Delta_{\bar{1}} f_1}{f} + (d-1)\frac{g_M(\nabla f,\nabla f)}{f^2})$,

where, analogous to *Proposition 1.1*, we consider $g_M(\nabla f, \nabla f) = \bar{g}_{B_1}(\nabla f_1, \nabla f_1) + \bar{g}_{B_2}(\nabla f_2, \nabla f_2)$.

Let φ_{1,x_ix_j}, φ_{1,x_i}, f_{1,x_ix_j}, f_{1,x_i}, φ_{2,y_ly_r}, φ_{2,y_l}, f_{2,y_ly_r} and f_{2,y_l}, be the second and the first order derivatives of φ_1, φ_2, f_1 and f_2, respectively, with respect to x_ix_j and y_ly_r.

Now we have:

(2.18) $Hess_1(\varphi_1)(X_i,X_j) = \varphi_{1,x_ix_j}$,

(2.19) $\Delta_1(\varphi_1) = \sum_{k=1}^{n_1} \varepsilon_k \varphi_{1,x_kx_k}$,

(2.20) $|\nabla_1(\varphi_1)|^2 = \sum_{k=1}^{n_1} \varepsilon_k \varphi_{1,x_k}^2$,

(2.21) $Hess_2(\varphi_2)(Y_l,Y_r) = \varphi_{2,y_ly_r}$,

(2.22) $\Delta_2(\varphi_2) = \sum_{s=1}^{n^2} \varepsilon_s \varphi_{2,y_ly_r}$

(2.23) $|\nabla_2(\varphi_2)|^2 = \sum_{s=1}^{n_2} \varepsilon_s \varphi_{2,y_s}^2$.

(2.24) $Hess_{\bar{1}}(f_1)(X_i,X_j) = f_{1,x_ix_j} - \sum_k \bar{\Gamma}_{ij}^k f_{1,x_k}$,

where $\bar{\Gamma}_{ij}^k = 0$, $\bar{\Gamma}_{ij}^i = -\frac{\varphi_{1,x_j}}{\varphi_1}$, $\bar{\Gamma}_{ii}^k = \varepsilon_i\varepsilon_k \frac{\varphi_{1,x_k}}{\varphi_1}$ and $\bar{\Gamma}_{ii}^i = -\frac{\varphi_{1,x_j}}{\varphi_1}$, so (2.24) becomes:

(2.25) $Hess_{\bar{1}}(f_1)(X_i,X_j) = f_{1,x_ix_j} + \frac{\varphi_{1,x_j}}{\varphi_1} f_{1,x_i} + \frac{\varphi_{1,x_i}}{\varphi_1} f_{1,x_j}$, for $i \neq j$, and

(2.26) $Hess_{\bar{1}}(f_1)(X_i,X_i) = f_{1,x_ix_i} + 2\frac{\varphi_{1,x_i}}{\varphi_1} f_{1,x_i} - \varepsilon_i \sum_{k=1}^{n_1} \varepsilon_k \frac{\varphi_{1,x_k}}{\varphi_1} f_{1,x_k}$.

Since $Hess_{\bar{2}}(f_2)(Y_l,Y_r) = 0$, we get:

(2.27) $Hess_{\bar{2}}(f_2)(Y_l,Y_r)=f_{2,y_ly_r}+\frac{\varphi_{2,y_r}}{\varphi_2}f_{2,y_l}+\frac{\varphi_{2,y_l}}{\varphi_2}f_{2,y_r}=0$, for $l\neq r$, and

(2.28) $Hess_{\bar{2}}(f_2)(Y_l,Y_l)=f_{2,y_ly_l}+2\frac{\varphi_{2,y_l}}{\varphi_2}f_{2,y_l}-\varepsilon_l\sum_{s=1}^{n_2}\varepsilon_s\frac{\varphi_{2,y_s}}{\varphi_2}f_{2,y_s}=0.$

Then the Ricci tensors are:

(2.29) $Ric_{\bar{B}_1}(X_i,X_j)=\frac{(n_1-2)\varphi_{1,x_ix_j}}{\varphi_1}$, for $i\neq j$,

(2.30) $Ric_{\bar{B}_1}(X_i,X_i)=\frac{(n_1-2)\varphi_{1,x_ix_i}+\varepsilon_i\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_kx_k}}{\varphi_1}-(n_1-1)\varepsilon_i\sum_{k=1}^{n_1}\frac{\varepsilon_k\varphi_{1,x_k}^2}{\varphi_1^2}$,

(2.31) $Ric_{\bar{B}_2}(Y_l,Y_r)=\frac{(n_2-2)\varphi_{2,y_ly_r}}{\varphi_2}$, for $l\neq r$,

(2.32) $Ric_{\bar{B}_2}(Y_l,Y_l)=\frac{(n_2-2)\varphi_{2,y_ly_l}+\varepsilon_l\sum_{s=1}^{n_2}\varepsilon_s\varphi_{2,y_sy_s}}{\varphi_2}-(n_2-1)\varepsilon_l\sum_{s=1}^{n_2}\frac{\varepsilon_s\varphi_{2,y_s}^2}{\varphi_2^2}$.

Using (2.29) and (2.25) in the (2.12) and then using (2.30) and (2.26) in the (2.12) we obtain respectively:

(2.33) $Ric_M(X_i,X_j)=\frac{(n_1-2)\varphi_{1,x_ix_j}}{\varphi_1}-\frac{d}{f}[f_{1,x_ix_j}+\frac{\varphi_{1,x_j}}{\varphi_1}f_{1,x_i}+\frac{\varphi_{1,x_i}}{\varphi_1}f_{1,x_j}]$, for $i\neq j$,

(2.34) $Ric_M(X_i,X_i)=\frac{(n_1-2)\varphi_{1,x_ix_i}+\varepsilon_i\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_kx_k}}{\varphi_1}-(n_1-1)\varepsilon_i\sum_{k=1}^{n_1}\frac{\varepsilon_k\varphi_{1,x_k}^2}{\varphi_1^2}+$

$$-\frac{d}{f}[f_{1,x_ix_i}+2\frac{\varphi_{1,x_i}}{\varphi_1}f_{1,x_i}-\varepsilon_i\sum_{k=1}^{n_1}\varepsilon_k\frac{\varphi_{1,x_k}}{\varphi_1}f_{1,x_k}],$$

while, using (2.31) and (2.27) in the (2.13) and then using (2.32) and (2.28) in the (2.13) we obtain respectively:

(2.35) $Ric_M(Y_l,Y_r)=\frac{(n_2-2)\varphi_{2,y_ly_r}}{\varphi_2}$, for $l\neq r$,

(2.36) $Ric_M(Y_l,Y_l)=\frac{(n_2-2)\varphi_{2,y_ly_l}+\varepsilon_l\sum_{s=1}^{n_2}\varepsilon_s\varphi_{2,y_sy_s}}{\varphi_2}-(n_2-1)\varepsilon_l\sum_{s=1}^{n_2}\frac{\varepsilon_s\varphi_{2,y_s}}{\varphi_2^2}$.

Now considering:

(2.37) $Ric_F=0$,

(2.38) $g_M(U_i,U_j)=f^2g_F(U_i,U_j)$, with $f=f_1+f_2$,

(2.39) $\Delta_{\bar{2}}(f_2)=0$

(2.40) $\Delta_{\bar{1}}(f_1)=\varphi_1^2\sum_{k=1}^{n_1}\varepsilon_kf_{1,x_kx_k}-(n_1-2)\varphi_1\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_k}f_{1,x_k}$,

(2.41) $g_M(\nabla f,\nabla f)=\varphi_1^2\sum_{k=1}^{n_1}\varepsilon_kf_{1,x_k}^2+\varphi_2^2\sum_{s=1}^{n_2}\varepsilon_sf_{2,y_s}^2$,

and by replacing them in (2.17):

(2.42) $Ric_M(U_i,U_j)=\{-f\varphi_1^2\sum_{k=1}^{n_1}\varepsilon_kf_{1,x_kx_k}+(n_1-2)f\varphi_1\sum_{k=1}^{n_1}\varepsilon_k\varphi_{1,x_k}f_{1,x_k}+$

$$-(d-1)(\varphi_1^2\sum_{k=1}^{n_1}\varepsilon_kf_{1,x_k}^2+\varphi_2^2\sum_{s=1}^{n_2}\varepsilon_sf_{2,y_s}^2)\}g_F(U_i,U_j).$$

Using the equations (2.33), (2.34), (2.35), (2.36) and (2.42), it follows that (M,g_M) is an Einstein manifold if and only if, the equations *(I), (II), (III), (IV), (V)* are satisfied. □

3. The positive constant Ricci curvature case ($\lambda>0$)

In this section we look for the existence of a solution to the positive constant Ricci curvature case ($\lambda > 0$) when the base-manifold is the product of two conformal manifolds to a n_1-dimensional and n_2-dimensional pseudo-Euclidean space, respectively, invariant under the action of a (n_1-1)-dimensional group of transformations and that the fiber F is flat.

Theorem 3.1: *Let (B, g_B), be the base-manifold $B = (B_1 \times B_2)$, $B_1 = \mathbb{R}^{n_1}$, with coordinates $(x_1, x_2, ..x_{n_1})$, $B_2 = \mathbb{R}^{n_2}$, with coordinates $(y_1, y_2, ..y_{n_2})$, where $n_1, n_2 \geq 3$, and let $g_B = g_{B_1} + g_{B_2}$ be the metrics on B, where $g_{B_1} = \varepsilon_i \delta_{ij}$ and $g_{B_2} = \varepsilon_l \delta_{lr}$.*
Let $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, $f_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, $\varphi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^{n_2} \to \mathbb{R}$, be smooth functions $f_1(\xi_1)$, $f_2(\xi_2)$, $\varphi_1(\xi_2)$ and $\varphi_2(\xi_2)$, such that $f(\xi_1, \xi_2) = f_1(\xi_1) + f_2(\xi_2)$ be as in Definition 1.1, where $\xi_1 = \sum_{i=1}^{n_1} \alpha_i x_i$, $\alpha_i \in \mathbb{R}$, and $\sum_i \varepsilon_i \alpha_i^2 = \varepsilon_{i0}$ or $\sum_i \varepsilon_i \alpha_i^2 = 0$, and by the same token $\xi_2 = \sum_{l=1}^{n_2} \alpha_l y_l$, $\alpha_l \in \mathbb{R}$, and $\sum_l \varepsilon_l \alpha_l^2 = \varepsilon_{l0}$ or $\sum_l \varepsilon_l \alpha_l^2 = 0$.
Finally, let (M, g_M) be $((B_1 \times B_2) \times_{f=f_1+f_2} F, g_M)$, with $g_M = \bar{g}_B + (f_1+f_2)^2 g_F$, with conformal metric $\bar{g}_B = \bar{g}_{B_1} + \bar{g}_{B_2}$, where $\bar{g}_{B_1} = \frac{1}{\varphi_1^2} g_{B_1}$, $\bar{g}_{B_2} = \frac{1}{\varphi_2{}^2} g_{B_2}$, and $F = \mathbb{R}^d$ with $g_F = -\delta_{ab}$.
Then, whenever $\sum_i \varepsilon_i \alpha_i^2 = \varepsilon_{i0}$ (and $\sum_l \varepsilon_l \alpha_l^2 = \varepsilon_{l0}$), the warped-product metric $g_M = \bar{g}_B + (f_1+f_2)^2 g_F$ is Einstein with constant Ricci curvature λ if and only if the functions f_1, f_2, φ_1 and φ_2 satisfy the following conditions:

(Ia) $(n_1-2) f \varphi_1'' - \varphi_1 f_1'' d - 2\varphi_1' f_1' d = 0$, *for* $i \neq j$,

(IIa) $\varphi_2'' = 0$, *for* $l \neq r$,

(IIIa) $\sum_k \varepsilon_k \alpha_k^2 [f \varphi_1 \varphi_1'' - (n_1-1) f \varphi_1'^2 + \varphi_1 \varphi_1' f_1' d] = \lambda f$,

(IVa) $\sum_s \varepsilon_s \alpha_s^2 [-(n_2-1) \varphi_2'^2] = \lambda$

(Va)
$$\sum_k \varepsilon_k \alpha_k^2 [-f \varphi_1^2 f_1'' + (n_1-2) f \varphi_1 \varphi_1' f_1' - (d-1) \varphi_1^2 f_1'^2] + \\ - \sum_s \varepsilon_s \alpha_s^2 [(d-1) \varphi_2^2 f_2'^2] = \lambda f^2.$$

Proof. We have:
$$\varphi_{1,x_i x_j} = \varphi_1'' \alpha_i \alpha_j, \quad \varphi_{1,x_i} = \varphi_1' \alpha_i, \quad f_{1,x_i x_j} = f_1'' \alpha_i \alpha_j, \quad f_{1,x_i} = f_1' \alpha_i,$$
and
$$\varphi_{2,y_l y_r} = \varphi_2'' \alpha_l \alpha_r, \quad \varphi_{2,y_l} = \varphi_2' \alpha_l, \quad f_{2,y_l y_r} = f_2'' \alpha_l \alpha_r, \quad f_{2,y_l} = f_2' \alpha_l.$$

Substituting these in *(I)* and *(II)* and if $i \neq j$ and $l \neq r$ such that $\alpha_i\alpha_j \neq 0$ and $\alpha_l\alpha_r \neq 0$, we obtain *(Ia)* and *(IIa)*.
In the same manner for *(III)* and *(IV)*, by considering the relation between φ_1'' and f_1'' from *(Ia)* and $\varphi_2'' = 0$ from *(IIa)*, we get *(IIIa)* and *(IVa)* respectively. Analogously, the equation *(V)* reduces to *(Va)*. □

Now we are going to look for the existence of a solution to the positive constant Ricci curvature case ($\lambda > 0$), considering $f_2(\xi_2) = 1$, and $dim(B_1) = dim(F)$, i.e., $n_1 = d$. So, whenever $\sum_{i=1}^{n_1} \alpha_i^2\varepsilon_i \neq 0$, without loss of generality, we may consider $\sum_{i=1}^{n_1} \alpha_i^2\varepsilon_i = -1$ (the same for $\sum_{l=1}^{n_2} \alpha_l^2\varepsilon_l \neq 0$, in which we consider $\sum_{l=1}^{n_2} \alpha_l^2\varepsilon_l = -1$).
In this way the equations *(Ia)*, *(IIa)*, *(IIIa)*, *(IVa)* *(Va)* become:

(Ib) $(n_1 - 2)(f_1 + 1)\varphi_1'' - n_1\varphi_1 f_1'' - 2n_1\varphi_1' f_1' = 0$, for $i \neq j$,

(IIb) $\varphi_2'' = 0$, for $l \neq r$,

(IIIb) $-(f_1 + 1)\varphi_1\varphi_1'' + (n_1 - 1)(f_1 + 1)\varphi_1'^2 - n_1\varphi_1\varphi_1' f_1' = \lambda(f_1 + 1)$,

(IVb) $(n_2 - 1)\varphi_2'^2 = \lambda$,

(Vb) $(f_1 + 1)\varphi_1^2 f_1'' - (n_1 - 2)(f_1 + 1)\varphi_1\varphi_1' f_1' + (n_1 - 1)\varphi_1^2 f_1'^2 = \lambda(f_1 + 1)^2$.

Note that since $f_2(\xi_2) = constant$, then the equations (2.27) and (2.28), concerning the condition $Hess_{\bar{2}}(f_2) = 0$, are obviously satisfied.
It is worth noticing that there is no reason to believe that any nontrivial solutions exist, since the system is overdetermined. One must first check out the compatibility conditions and fortunately this is easy to figure out. Changing the notation: from $\big(\xi_1, \varphi_1(\xi_1), f_1(\xi_1)\big)$, to $\big(t, \beta(t), \gamma(t){-}1\big)$ (in order to simplify the writing and avoid confusion with the indexes), and also writing $\lambda = qm^2/2 > 0$, where $q = n_1$, i.e. $dim(B_1)$, our system of equations then becomes:

$$(3.1)\begin{cases} (q-2)\gamma\beta'' - q\beta\gamma'' - 2q\beta'\gamma' = 0 \\ -\beta\gamma\beta'' - (q-1)\gamma\beta'^2 - q\beta'\gamma' - \frac{1}{2}qm^2\gamma = 0 \\ \gamma\beta^2\gamma'' - (q-2)\beta\gamma\beta'\gamma' + (q-1)\beta^2\gamma'^2 - \frac{1}{2}qm^2\gamma^2 = 0 \end{cases}$$

So, if we solve the second and third equations for β'' and γ'' and substituting them into the first equation, we note that the first equation can be replaced by a first order equation, that is:

(3.2) $(q-2)\gamma^2\beta'^2 - 2q\beta\gamma\beta'\gamma' + q\beta^2\gamma'^2 - qm^2\gamma^2 =: Z(\beta, \gamma, \beta', \gamma') = 0.$

Now, differentiating Z with respect to t and then eliminating β'' and γ'' using the second and third equations of (3.1), the resulting expression in $(\beta, \gamma, \beta', \gamma')$ is a multiple of $Z(\beta, \gamma, \beta', \gamma')$. This shows us that the combined system of equations (3.1) and (3.2) satisfies the compatibility conditions, so that the system has solutions, specifically, a 3-parameter family of them.

If we want to describe these solutions more explicitly, we must note that the equations are t-autonomous and have a 2-parameter family of scaling symmetries. In particular, the equations are invariant under the 3-parameter group of transformations of the form:

(3.3) $\Phi_{a,b,c}(t, \beta, \gamma) = (at+c, a\beta, b\gamma)$

where a and b are nonzero constants and c is any constant. In fact, the equation (3.2) implies that there is a function $\omega(t)$ such that

(3.4) $\begin{cases} \beta' = \frac{2mq\omega(\omega-1)}{\left((q-2)\omega^2-2q\omega+q\right)} \\ \gamma' = \frac{m\gamma\left((q-2)\omega^2-q\right)}{\beta\left((q-2)\omega^2-2q\omega+q\right)} \end{cases}$

and then the second and third equations of (3.1) imply that ω must satisfy

(3.5) $\omega' = \frac{m\left(q+2q\omega-(3q-2)\omega^2\right)}{\beta}.$

Conversely, the combined system of (3.4) and (3.5) gives the general solution of the original system. This latter system is easily integrated by the usual separation of variables method, i.e., by eliminating t yields a system of the form:

(3.6) $\frac{d\beta}{\beta} = R(\omega)d\omega$

and

(3.7) $\frac{d\gamma}{\gamma} = S(\omega)d\omega$

where $R(\omega)$ and $S(\omega)$ are rational functions of ω. Writing β and γ as elementary functions of ω, then we can also write:

(3.8) $dt = \beta T(\omega) d\omega,$

where T is a rational function of ω, so that t can be written as a function of ω by quadrature. Thus, we have the integral curves in $(t, \beta, \gamma, \omega)$-space in terms of explicit functions.

In conclusion (because of the 3-parameter family of equivalences of solutions), we can say that in certain sense, these solutions are all equivalent to a finite number of possibilities.

Remarks: As is well known, an Einstein warped product manifold with Riemannian-metric and Ricci-flat fiber-manifold can only admit zero or negative Ricci tensor, $Ric \leq 0$. Here we have shown, that a simple pseudo-Riemannian metric construction allows, an Einstein warped product manifold with Ricci-flat fiber-manifold, to obtain $Ric > 0$, and this may find interest, for example, in how to build warped-product spacetime models, with positive curvature, whose fiber is Ricci-flat.

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