Uniformization Problems on Subvarieties of Bounded Symmetric Domains and Their Quotient Manifolds

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Abstract Motivated by the Uniformization Theorem for Riemann surfaces, uniformization problems play an important role in Kähler geometry. In the study of negatively curved compact Kähler manifolds, bounded symmetric domains Ω are the analogues of the unit disk, and we have the well-known strong rigidity theorem discovered by Siu on compact Kähler manifolds homotopic to compact quotients of irreducible bounded symmetric domains of dimension ≥ 2. In this lecture we are first of all concerned with algebraic subvarieties of bounded symmetric domains admitting compact quotients by global automorphisms. As it turns out such problems and their generalizations are also studied in arithmetic geometry in view of their links to various conjectures on special points and special subvarieties. We will explain how uniformization theorems for bi-algebraic varieties can be proven by analytic methods involving the Poincaré-Lelong equation in the cocompact case (joint work with S.-T. Chan), generalizing in the cocompact case earlier results of Ullmo-Yafaev (2011) in the case of arithmetic quotients. In contrast to earlier results, for our method it suffices to consider an algebraic subvariety $Z \subset \Omega$ for which there exists a (torsion-free) discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ leaving Z invariant such that $Z_{\check{\Gamma}} := Z/\check{\Gamma} \subset \Omega/\check{\Gamma} =: X_{\check{\Gamma}}$ is compact, and we prove that $Z_{\check{\Gamma}} \subset X_{\check{\Gamma}}$ is totally geodesic. (In the arithmetic setting $\check{\Gamma} \subset \Gamma$, possibly of infinite index, such that $\Gamma \subset \mathrm{Aut}(\Omega)$ is an arithmetic lattice.) We will also explore the problem of characterization of Zariski closures $\mathscr{Z}$ of images of algebraic subsets $S \subset \Omega$ under the uniformization map $\pi : \Omega \to \Omega/\Gamma =: X_\Gamma$ when $\Gamma \subset \mathrm{Aut}(\Omega)$ is an arbitrary lattice and give a proof in the case where Ω is the complex unit ball $\mathbb{B}^n$, $n \geq 2$. (For arithmetic lattices it is a theorem of Klingler-Ullmo-Yafaev (2016) that $\mathscr{Z} \subset X_\Gamma$ is weakly special, i.e., totally geodesic.) Our proof in the rank-1 case makes use of of foliation theory, Chow schemes and Kähler geometry.