LCK structures with holomorphic Lee vector field

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Conformal Structures in Geometry

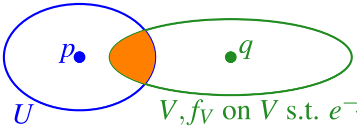
Zoom Conference for Liviu Ornea's 60th birthday

July 16, 2020

Context and Notation: LCK structures

Let (M^{2n}, J) be a complex manifold of complex dimension $n \geq 2$.

A Hermitian metric g is *locally conformally Kähler (lcK)* if



V, f_V on V s.t. $e^{-f_V} g|_V$ is Kähler

$\exists\, f_U$ on U s.t. $e^{-f_U} g|_U$ is Kähler

$$\mathrm{d}f_U|_{U\cap V} = \mathrm{d}f_V|_{U\cap V} =: \theta \ \text{(Lee form)}$$

$$\boxed{\mathrm{d}\Omega = \theta \wedge \Omega} \ (\text{for } \Omega := g(J\cdot, \cdot))$$

<u>Notation</u>:

- (g, Ω, θ) is an lcK structure on (M, J). Assume $\theta \neq 0$ (i.e. g is not Kähler).
- $T := \theta^{\sharp_g}$ (Lee vector field)

Special case: Vaisman structures

An lcK structure (g, Ω, θ) is called Vaisman if θ is parallel: $\nabla^g \theta = 0$.

Some properties of Vaisman structures:

- A Vaisman structure on (M, J) is (up to scaling) uniquely determined by θ:

$$\boxed{\Omega = \frac{1}{|\theta|_g^2}(\theta \wedge J\theta - \mathrm{d}J\theta)}$$

- T and JT are commuting holomorphic Killing vector fields.
- If M is **compact** and of Vaisman-type, then:
  - all Vaisman structures have the same Lee vector field (up to $\mathbb{R}^+$).
  - the Lee forms of all Vaisman structures form a convex cone in $\Omega^1(M)$.

Notation:

- $\mathbb{R}^+ T_0$ denotes the space of Lee vector fields of all Vaisman structures.
- Assume that Vaisman structures (g, Ω, θ) are *normalized*: $|\theta|_g = 1$.

LCK structures with holomorphic Lee vector field

Let (M, J) be a **compact** complex manifold.
Let (g, Ω, θ, T) be an lcK structure on (M, J) with T **holomorphic**.

Previous results:

- **A. Moroianu, S. Moroianu, L. Ornea**:
  - $|T|_g$ is constant $\Longrightarrow (g, \Omega, \theta, T)$ is Vaisman.
  - T is divergence-free $\Longrightarrow (g, \Omega, \theta, T)$ is Vaisman.
  - examples of non-Vaisman (g, Ω, θ, T) with T holomorphic (on (M, J) of Vaisman-type and $T \in \mathbb{R}^+ T_0$).
- **N. Istrati**: $\Omega = a(\theta \wedge J\theta - \mathrm{d}J\theta), a \in \mathbb{R}^* \Longrightarrow (g, \Omega, \theta, T)$ is Vaisman.
- **F. Belgun**: examples of (g, Ω, θ, T) with T holomorphic on (M, J) which is not of Vaisman-type.

Our goal: on a **compact manifold of Vaisman-type**:

Which holomorphic vector fields occur as Lee fields of lcK structures?

Which are these lcK structures?

Holomophic Lee vector fields on Vaisman-type manifolds

Let (M,J) be a **compact** complex manifold of **Vaisman-type**.

Let $\mathbb{R}^+ \boldsymbol{T_0}$ be the space of Lee vector fields of Vaisman structures on (M,J).

Let (g,Ω,θ,T) be an **lcK structure** on (M,J) with $\boldsymbol{T}$ **holomorphic**.

Remark JT is Killing w.r.t. g (because $\mathcal{L}_{JT}\Omega = 0$ by Cartan's formula).

$$\boxed{\text{How is } \boldsymbol{T} \text{ related to } \mathbb{R}^+\boldsymbol{T_0}?}$$

Theorem (F. Madani, A. Moroianu, -)

There exists an **adapted Vaisman structure** (g_0,Ω_0,θ_0) *on* (M,J)*, i.e.*

$$[\theta_0] = [\theta] \text{ and } \theta_0(JT) = 0$$

and a **positive** *function* $\boldsymbol{h} \in \mathcal{C}^\infty(M)$ *s.t.*

$$\boxed{\boldsymbol{T} = \boldsymbol{h}\boldsymbol{T_0} - \mathrm{grad}^{g_0}\boldsymbol{h}.}$$

In particular, $\mathcal{L}_{JT}\theta_0 = 0$*, so* JT *is holomorphic Killing on* (M,J,g_0).

1^{st} Step: Existence of an adapted Vaisman structure

Given (g, Ω, θ, T) an lcK structure with T holomorphic.

(1) There exists a Vaisman structure $(g_1, \Omega_1, \theta_1)$ with $[\theta_1] = [\theta]$.
(→ use so-called deformations of type I of Vaisman structures)

(2) Define by an **averaging process**:

$$\boxed{\theta_0 := \int_{\gamma \in G} \gamma^* \theta_1 \, \mathrm{d}\mu \in \Omega^1(M)}$$

where $G := \overline{\{\varphi_t\}}$ (the closure of the flow φ_t of JT, Killing w.r.t. g) is a compact torus in $\mathrm{Iso}(M, g)$ with Haar measure $\mathrm{d}\mu$.Then:

- $[\theta_0] = [\theta_1]$.
- $\mathcal{L}_{JT}\theta_0 = 0$ ($\Rightarrow \theta_0(JT) = 0$, by Cartan's formula and M-compact).
- θ_0 defines a Vaisman structure.

(→ the set of Lee forms of Vaisman structures is a convex cone)

2^{nd} Step: Holomorphic Killing vector fields of Vaisman str.

Proposition

Let $(M, J, g_0, \Omega_0, \theta_0)$ *be* **compact Vaisman** *with Lee vector field* T_0. *Then any holomorphic Killing vector field* K *on* (M, J, g_0) *is of the form:*

$$K = cT_0 + hJT_0 + K_0$$

where:

- $K_0 \in \{T_0, JT_0\}^{\perp}$
- c *is a constant*
- $h \in \mathcal{C}^{\infty}(M)$ *satisfies:*

$$K_0 \lrcorner \Omega_0 = \mathrm{d}h \text{ and } T_0(h) = JT_0(h) = 0.$$

The function h *is called the Hamiltonian of* K.

3^{rd} Step: The positivity of the Hamiltonian

- T is nowhere vanishing
- $[\theta] = [\theta_0] \Longrightarrow \theta = \theta_0 + \mathrm{d}f$ and:

$$|T|_g^2 = h + hT_0(f) - \langle \mathrm{d}f, \mathrm{d}h\rangle_{g_0} > 0$$

- if $m := \min_M h$, then we obtain at $p \in h^{-1}(m)$:

$$m(1 + T_0(f)(p)) > 0 \Longrightarrow m \neq 0$$

- assuming $m < 0$, implies $T_0(f)(p) < -1$, for all $p \in h^{-1}(m)$
  T_0 parallel w.r.t. $g_0 \Longrightarrow \gamma :=$integral curve of T_0 is complete geodesic

$$T_0(f)(\gamma(t)) \overset{T_0(h)=0}{=} T_0(f)(p) < -1, \quad \forall t \in \mathbb{R} \quad \lightning$$

f is bounded

$$\Longrightarrow m > 0$$

Holomophic Lee vector fields on Vaisman-type manifolds

Theorem (F. Madani, A. Moroianu, -)

Let (M, J) be a **compact** *complex manifold of* **Vaisman-type**.

If (g, Ω, θ, T) is an **lcK structure** *on (M, J) with* $\boldsymbol{T}$ **holomorphic**, *then there exists an* **adapted Vaisman structure** $(g_0, \Omega_0, \theta_0)$ *on (M, J), i.e.*

$$[\theta_0] = [\theta] \text{ and } \theta_0(JT) = 0$$

and a **positive** *function* $\boldsymbol{h} \in \mathcal{C}^\infty(M)$ *s.t.*

$$\boxed{\boldsymbol{T} = \boldsymbol{h}\boldsymbol{T_0} - \mathrm{grad}^{\boldsymbol{g_0}}\boldsymbol{h}.}$$

In particular, $\mathcal{L}_{JT}\theta_0 = 0$, so JT is holomorphic Killing on (M, J, g_0).

Conversely, if (g_0, θ_0, T_0) is a Vaisman structure and K a holomorphic Killing vector field on (M, J, g_0) with positive Hamiltonian and vanishing T_0-component, then $-JK$ is the Lee vector field of an lcK structure.

The space of holomorphic Lee vector fields

$\mathcal{HL}(M,J) :=$ set of holomorphic vector fields which occur as Lee fields of some lcK structure on (M,J) - **compact of Vaisman-type**.

- **Description** of $\mathcal{HL}(M,J)$ **in terms of Vaisman structures**:

$$\mathcal{HL}(M,J) = \left\{ T := hT_0 - \mathrm{grad}^{g_0} h \;\middle|\; \begin{array}{c} (g_0, \Omega_0, \theta_0, T_0) \text{ is Vaisman}, \\ JT \text{ is holomorphic and Killing w.r.t. } g_0, \\ h \in \mathcal{C}^\infty(M, \mathbb{R}^+) \end{array} \right\}$$

- **Intrinsic description** of $\mathcal{HL}(M,J)$:

Theorem (F. Madani, A. Moroianu, -)

A holomorphic vector field T belongs to $\mathcal{HL}(M,J)$ if and only if the following two conditions are satisfied:
(i) JT is of Killing-type.
(ii) $\forall p \in M$: $T(p) \in \mathbb{R}T_0(p) \oplus \mathbb{R}JT_0(p) \Longrightarrow T(p) \in \mathbb{R}^+ T_0(p)$.

La Mulți Ani, Liviu!