Examples emerging from LCK geometry

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Summary

Oeljeklaus-Toma manifolds:

- generalizing Inoue-Bombieri
- construction based on algebraic number theory
- complex non-Kähler manifolds
- LCK metrics

K. Oeljeklaus, M. Toma, Non-Kähler compact complex manifolds associated to number fields, *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 1, 161–171.

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Summary

Oeljeklaus-Toma manifolds:

- generalizing Inoue-Bombieri
- construction based on algebraic number theory
- complex non-Kähler manifolds
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We prove:

- line bundles on OT are flat
- OT of simple-type are rigid under complex deformations

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- k algebraic number field,
  i.e. $k \simeq \mathbb{Q}[x]/(f)$ for $f \in \mathbb{Q}[x]$ monic of degree $n = [k : \mathbb{Q}]$
- $\rightsquigarrow$ it admits $n = s + 2t$ embeddings:

$$\sigma_1, \ldots, \sigma_s \colon k \to \mathbb{R},$$

$$\sigma_{s+1}, \ldots, \sigma_{s+t}, \bar{\sigma}_{s+1}, \ldots, \bar{\sigma}_{s+t} \colon k \to \mathbb{C}$$

- $\mathcal{O}_k$:= ring of algebraic integers has $\mathrm{rk}_{\mathbb{Z}}\,\mathcal{O}_k = s + 2t$
  Define $\mathcal{O}_k \circlearrowleft \mathbb{C}^{s+t}$ by translations

$$T_a(z_1, \ldots, z_{s+t}) := (z_1 + \sigma_1(a), \ldots, z_{s+t} + \sigma_{s+t}(a))$$

- $\mathcal{O}_k^\times$:= group of units has $\mathrm{rk}_{\mathbb{Z}}\,\mathcal{O}_k^\times = s + t - 1$
  Define $\mathcal{O}_k^\times \circlearrowleft \mathbb{C}^{s+t}$ by rotations

$$R_u(z_1, \ldots, z_{s+t}) := (\sigma_1(u) \cdot z_1, \ldots, \sigma_{s+t}(u) \cdot z_{s+t})$$

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- Consider

$$\mathcal{O}_k \rtimes \mathcal{O}_k^\times \circlearrowleft \mathbb{C}^{s+t}$$

- $\mathcal{O}_k \rtimes \mathcal{O}_k^\times \circlearrowleft \mathbb{C}^{s+t}$ has fixed points on $\mathbb{R}^s \times \mathbb{C}^t$
  $\rightsquigarrow$ Consider
$$\mathcal{O}_k \rtimes \mathcal{O}_k^{\times,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$$
  where $\mathcal{O}_k^{\times,+}$:= subgroup of totally-positive units

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- $\mathcal{O}_k \rtimes \mathcal{O}_k^{\times,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$ is free but not properly-discontinuous:

$$\mathrm{rk}_{\mathbb{Z}}\, \mathcal{O}_k + \mathrm{rk}_{\mathbb{Z}}\, \mathcal{O}_k^{\times,+} = (s+2t) + (s+t-1) \geq 2(s+t)$$

Thm (Oeljeklaus, Toma)

For any s, t, there exists $U \subseteq \mathcal{O}_k^{\times,+}$ of $\mathrm{rk}_{\mathbb{Z}} U = s$ such that

$$\mathcal{O}_k \rtimes U \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$$

is fixed-point-free, properly-discontinuous, co-compact.
Set

$$X(k, U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_k \rtimes U$$

Say $X(k, U)$ *simple-type* if there is no intermediate extension $\mathbb{Q} \subset k' \subset k$ compatible with U

Oeljeklaus-Toma manifolds

properties

- $s = t = 1$ gives Inoue-Bombieri surface
- $\mathrm{Kod} = -\infty$, $b_1 = s$, $b_2 = \binom{s}{2}$ [OT; Istrati-Otiman]
- Dolbeault cohomology and Frölicher [Tomassini-Torelli; Otiman-Toma]
- LCK for $t = 1$, non-lcK for $s < t$, non-Vaisman [Tricerri; OT; Vuletescu; Dubickas; Kasuya]
- OT are locally-homogeneous solvmanifolds [Kasuya]
- OT have no curves, no surfaces except Inoue [S. Verbitsky]
- Ω^1_X, Θ_X, $K_X^{\otimes \ell}$ are flat, no global sections [OT]

Line bundles on OT
flatness

Thm (DA, Parton, Vuletescu)

Any line bundle on Oeljeklaus-Toma manifold is flat.

Proof (continue...)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \mathbb{C}_X & \longrightarrow & \mathbb{C}_X^* & \longrightarrow & 0
\end{array}$$

Proof (continue…)

$$
\begin{array}{ccccccccc}
H^1(X;\mathbb{Z}_X) & \longrightarrow & H^1(X;\mathcal{O}_X) & \longrightarrow & H^1(X;\mathcal{O}_X^*) & \longrightarrow & H^2(X;\mathbb{Z}_X) & \longrightarrow & H^2(X;\mathcal{O}_X) \\
\| & & \uparrow m & & \uparrow n & & \| & & \uparrow q \\
H^1(X;\mathbb{Z}_X) & \longrightarrow & H^1(X;\mathbb{C}_X) & \longrightarrow & H^1(X;\mathbb{C}_X^*) & \longrightarrow & H^2(X;\mathbb{Z}_X) & \longrightarrow & H^2(X;\mathbb{C}_X)
\end{array}
$$

Proof (continue. . .)

$$\begin{array}{ccccccccc}
H^1(X;\mathbb{Z}_X) & \longrightarrow & H^1(X;\mathcal{O}_X) & \longrightarrow & H^1(X;\mathcal{O}_X^*) & \longrightarrow & H^2(X;\mathbb{Z}_X) & \longrightarrow & H^2(X;\mathcal{O}_X) \\
\| & & \uparrow m & & \uparrow n & & \| & & \uparrow q \\
H^1(X;\mathbb{Z}_X) & \longrightarrow & H^1(X;\mathbb{C}_X) & \longrightarrow & H^1(X;\mathbb{C}_X^*) & \longrightarrow & H^2(X;\mathbb{Z}_X) & \longrightarrow & H^2(X;\mathbb{C}_X)
\end{array}$$

Claim A $\dim H^1(X;\mathcal{O}_X) = s$

Claim B $q\colon H^2(X;\mathbb{C}) \to H^2(X;\mathcal{O}_X)$ injective

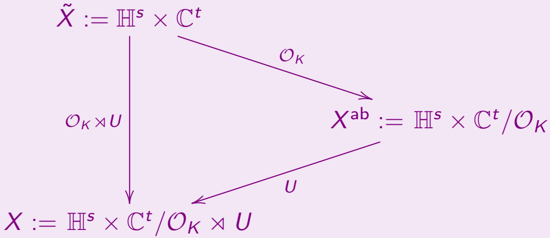
Line bundles on OT

flatness — proof

Proof (continue. . .)

$$\begin{array}{ccc}
\tilde{X} := \mathbb{H}^s \times \mathbb{C}^t & \xrightarrow{\mathcal{O}_K} & X^{\mathrm{ab}} := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \\
\Big\downarrow{\scriptstyle \mathcal{O}_K \rtimes U} & \swarrow{\scriptstyle U} & \\
X := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U & &
\end{array}$$

Proof (continue. . .)



Lyndon-Hochschild-Serre spectral seq of $0 \to \mathcal{O}_K \to \mathcal{O}_K \rtimes U \to U \to 0$ wrt $R = H^0(\tilde{X}; \mathcal{O}_{\tilde{X}})$:

$$0 \to \mathbb{C}^s \to H^1(X; \mathcal{O}_X) \to H^1(X^{ab}; \mathcal{O}_{X^{ab}})^U \to H^2(U; \mathbb{C}_U) \to H^2(X; \mathcal{O}_X)$$

Proof (continue...)

Claim A is consequence of

Claim A' $H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})^U = 0$

Proof (continue...)

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Claim A' $H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})^U = 0$

- $X^{\mathrm{ab}} \to \mathbb{C}^t/T$ hol fibre bundle with Stein fibres
- Borel-Serre: $H^{0,1}_{\bar\partial}(X^{\mathrm{ab}}) \simeq H^{0,1}_{\bar\partial}(B; \mathcal{O}_F)$
- take $[\alpha] = \left[\sum_j f_j(w,z) d\bar z_j\right]$ in $H^1(X^{\mathrm{ab}}; \mathcal{O}_{X^{\mathrm{ab}}})$
- we claim it has a flat representative $\sum_j c_j d\bar z_j \in [\alpha]$:
- we formally solve $\frac{\partial g}{\partial \bar z_j} = f_j - c_j$ by Fourier expansion
- regularity follows by Subspace Theorem

□

Proof (continue. . .)

Lyndon-Hochschild-Serre spectral seq of $0 \to \mathcal{O}_K \to \mathcal{O}_K \rtimes U \to U \to 0$ wrt $S = H^0(\tilde{X}; \mathbb{C}_{\tilde{X}})$:

$$0 \to \mathbb{C}^s \to H^1(X; \mathbb{C}_X) \to H^1(X^{\mathrm{ab}}; \mathbb{C}_{X^{\mathrm{ab}}})^U \to H^2(U; \mathbb{C}_U) \to H^2(X; \mathbb{C}_X)$$

Line bundles on OT

flatness — proof (claim B)

Proof (continue. . .)

Lyndon-Hochschild-Serre spectral seq of $0 \to \mathcal{O}_K \to \mathcal{O}_K \rtimes U \to U \to 0$ wrt $S = H^0(\tilde{X}; \mathbb{C}_{\tilde{X}})$:

$$0 \to \mathbb{C}^s \to H^1(X; \mathbb{C}_X) \to H^1(X^{\mathrm{ab}}; \mathbb{C}_{X^{\mathrm{ab}}})^U \to H^2(U; \mathbb{C}_U) \to H^2(X; \mathbb{C}_X)$$

The map $\mathbb{C}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$ induces

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{C}^s & \to & H^1(X;\mathbb{C}_X) & \xrightarrow[\text{[OT]}]{0} & H^1(X^{\mathrm{ab}};\mathbb{C}_{X^{\mathrm{ab}}})^U & \to & H^2(U;\mathbb{C}_U) & \to & H^2(X;\mathbb{C}_X) & \xrightarrow[\text{[OT], [IO]}]{} & 0\\
\| & & \| & & \downarrow & & \downarrow & & \| & & \downarrow q & & \\
0 & \to & \mathbb{C}^s & \to & H^1(X;\mathcal{O}_X) & \xrightarrow[0]{\text{Claim A'}} & H^1(X^{\mathrm{ab}};\mathcal{O}_{X^{\mathrm{ab}}})^U & \to & H^2(U;\mathbb{C}_U) & \to & H^2(X;\mathcal{O}_X) & & \\
 & & & & & & \downarrow{\scriptstyle \text{Claim A'}} & & & & & & \\
 & & & & & & 0 & & & & & &
\end{array}$$

Thm (APV)

For $X = X(K, U)$ Oeljeklaus-Toma manifold, for $\rho \colon U \to \mathbb{C}^*$ faithful representation with associated line bundle L_ρ, then $H^1(X; L_\rho) = 0$ unless $\rho = \bar{\sigma}_i^{-1}$ for some $i\{t+1, \dots t+s\}$.

Rigidity of OT

Cor (APV)

Oeljeklaus-Toma manifolds of simple-type are rigid under deformations of the complex structure.

Proof.

The holomorphic tangent bundle splits as

$$\Theta_X = \bigoplus_{j=1}^{s+t} L_{\sigma_j^{-1}},$$

so $H^1(X;\Theta_X) = 0$, unless $\sigma_i^{-1}(u) = \overline{\sigma}_j^{-1}(u)$ for some $i \in \{1, \ldots s+t\}$, $j \in \{s+1, \ldots, s+t\}$; but this does not happen if simple-type. □

Rmk

For $t = 1$: Braunling

Auguri Direttore!!

"Dieci anni era un traguardo solenne, per la prima volta si scriveva l'età con doppia cifra. L'infanzia smette ufficialmente quando si aggiunge il primo zero agli anni."