Kähler-like conditions and Vaisman metrics

Anna Fino

Dipartimento di Matematica
Universitá di Torino

"Conformal structures in Geometry",
in honour of Liviu Ornea's 60th birthday
16 July 2020

1. Kähler-like conditions

2. Link with Vaisman metrics on complex surfaces

Gauduchon connections

On any Hermitian manifold (M^{2n}, J, g) there exists an affine line of canonical Hermitian connections ∇^t ($\nabla^t J = 0$, $\nabla^t g = 0$), completely determined by their torsion

$$T(X, Y, Z) := g(T(X, Y), Z).$$

The family includes:

- the Chern connection ∇^{Ch} (T has trivial $(1,1)$-component)
- the Bismut (or Strominger) connection ∇^B (T is a 3-form)

Bismut and Chern connections

Remark

∇^{Ch} and ∇^{B} are related to the Levi-Civita connection ∇^{LC} by

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z),$$
$$g(\nabla^{Ch}_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z),$$

where $d^c = -J^{-1}dJ$ and ω is the associated fundamental form.

Remark

The trace of the torsion of ∇^{Ch} is equal to the Lee form $\theta := Jd^*\omega$, which is the unique 1-form satisfying

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}.$$

SKT metrics

$\nabla^B = \nabla^{LC} \Longleftrightarrow (M^{2n}, J, g)$ is Kähler

Definition

A Hermitian metric g on (M^{2n}, J) is said to be strong Kähler with torsion (SKT) or pluriclosed if $dT = 0$, i.e. if $\partial\overline{\partial}\omega = 0$.

Definition

A Hermitian metric ω is called Gauduchon if $dd^c\omega^{n-1} = 0$, or equivalently if $d^*\theta = 0$.

Remark

For $n = 2$ Gauduchon and SKT metrics coincide!

Kähler-like conditions

Remark

In general ∇^B does not satisfy the first Bianchi identity, since

$$\begin{aligned}\sigma_{X,Y,Z}\, R^B(X,Y,Z,U) &= dT^B(X,Y,Z,U) + (\nabla^B_U T^B)(X,Y,Z) \\ &\quad -\sigma_{X,Y,Z}\, g(T^B(X,Y), T^B(Z,U)).\end{aligned}$$

Definition

∇^B is Kähler-like if it satisfies the first Bianchi identity

$$\sigma_{X,Y,Z}\, R^B(X,Y,Z) = 0$$

and the type condition

$$R^B(X,Y,Z,W) = R^B(JX,JY,Z,W),\ \forall X,Y,Z,W.$$

Conjecture (Angella, Otal, Ugarte, Villacampa)

If for a Hermitian manifold (M^{2n}, J, g) the Bismut connection ∇^B is Kähler-like, then g is SKT.

• If ∇^B is flat and M is compact, then M admits as finite unbranched cover, a local Samelson space, given by the product of a compact semisimple Lie group and a torus [Q. Wang, B. Yang, F. Zheng].

• The conjecture is true for 6-dimensional compact solvmanifolds with holomorphically trivial canonical bundle [Angella, Otal, Ugarte, Villacampa].

Problem

Study the relations between the first Bianchi identity for ∇^B, the SKT condition and the parallelism of T^B.

Theorem (F, Tardini)

Let M^{2n} be a complex manifold with a SKT metric g.

- *If ∇^B satisfies the first Bianchi identity, then $\nabla^B T^B = 0$.*
- *If $\nabla^B T^B = 0$, then ∇^B satisfies the first Bianchi identity $\iff$ g is SKT.*

As a consequence:

Corollary (F, Tardini)

Let (M^{2n}, J, g) be a Hermitian manifold such that ∇^B satisfies the first Bianchi identity. Then

$$\nabla^B T^B = 0 \iff g \text{ is } SKT.$$

In relation to the Levi-Civita connection

Theorem (F, Tardini)

Let (M^{2n}, J, g) be a Hermitian manifold. If ∇^B satisfies the first Bianchi identity and g is SKT, then $\nabla^{LC} T^B = 0$.

Vaisman metrics

Definition

A Hermitian metric g on a complex manifold M^{2n} is a Vaisman metric if $d\omega = \theta \wedge \omega$, for some d-closed 1-form θ with $\nabla^{LC}\theta = 0$.

- Vaisman metrics are Gauduchon and $|\theta|$ is constant.
- If $n = 2$, then $T^B = - * \theta$.

Theorem (F, Tardini)

Let (M, J, g) be a Hermitian surface. Then, g is Vaisman if and only if g is SKT and ∇^B satisfies the first Bianchi identity.

Pluriclosed flow

Let (M^{2n}, J, g_0) be an Hermitian manifold. Streets and Tian introduced the flow

$$\frac{\partial \omega(t)}{\partial t} = -(\rho^B)^{1,1}(\omega(t)), \quad \omega(0) = \omega_0.$$

Theorem (Streets,Tian)

Let (M^{2n}, J) be a compact complex manifold. If ω_0 is SKT, then $\exists \epsilon > 0$ and a unique solution $\omega(t)$ to the pluriclosed flow with initial condition ω_0.
If ω_0 is Kähler, then $\omega(t)$ is the unique solution to the Kähler-Ricci flow with initial data ω_0.

Problem

Study the behaviour of the Vaisman condition along the pluriclosed flow.

Theorem (F, Tardini)

Let M be a compact complex surface admitting a Vaisman metric ω_0 with constant scalar curvature, then the pluriclosed flow starting with ω_0 preserves the Vaisman condition.

We use that, if (M, J, g) is a compact Vaisman surface, then $\rho^{Ch} = h\, dJ\theta$, for some $h \in \mathcal{C}^\infty$. Moreover, $Scal(g)$ is constant if and only if h is constant and, in such a case $c_1(M) = 0$.

HAPPY BIRTHDAY, LIVIU!

LA MULTI ANI, LIVIU!!