A_∞-ALGEBRE

ref.: – STASHEFF, Homotopy associativity of H-spaces, I, II, TRANS. AMS 108 (1963), 275-292, 293-312;
– KELLER, Introduction to A-infinity algebras and modules, HOMOL. HOMOT. APPL. 3 (2001), 1-35; arXiv: 9910179 [math.RA].
– GETZLER, JONES, A_∞ algebras and the cyclic bar complex, ILLINOIS J. MATH. 34 (1990), 256-283.

Introduzione

MOTIVAZIONE: quale struttura algebrica su $H^\bullet(A^\bullet, d)$ permette di ricostruire $(A^\bullet, d)$?
~~non basta alg.~~ $\rightsquigarrow$ A_∞-alg: si veda [KELLER; §4.3].

ex. X varietà C^∞ cpt, g metrica Riemanniana:

$\alpha, \beta \in \ker\Delta \rightsquigarrow \alpha\wedge\beta \in \ker d$

ma, in generale, $\ker\Delta$ non è un'algebra

$$[\alpha]\cup[\beta] = [\alpha\wedge\beta]$$

$$(\alpha^H + d\lambda_\alpha)\wedge(\beta^H + d\lambda_\beta) - d\lambda_{\alpha\wedge\beta} = (\alpha\wedge\beta)^H$$

Oggetti

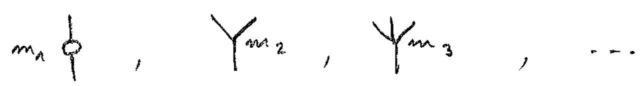
DEF Una A_∞-algebra$/k$ (detta anche strongly homotopy associative = sha algebra) è:
- $A = \bigoplus_{i\in\mathbb{Z}} A^i$ sp. vettoriale $\mathbb{Z}$-graduato,
- $\forall n\geq 1$, $m_n : A^{\otimes n}\to A$ mappa k-lineare omogenea di grado $2-n$

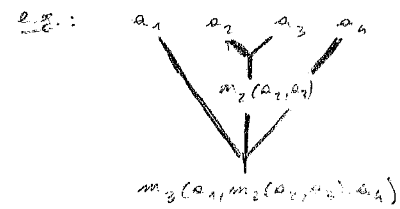
tc $$\boxed{\sum_{n=r+s+t} (-1)^{r+st}\, m_{r+1+t}\left(id^{\otimes r}\otimes m_s\otimes id^{\otimes t}\right) = 0}$$

RMK Usiamo la convenzione dei segni di KOSZUL.

$$(f\otimes g)(x\otimes y) = (-1)^{\deg g\cdot\deg x} f(x)\otimes g(y).$$

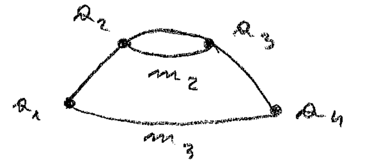
RMK Rappresentazione grafica:





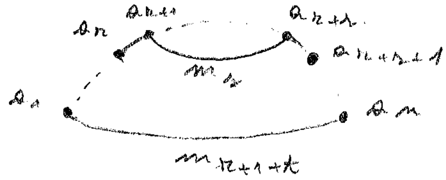
Rappresentazione grafica di Kontsevich:

$m_3(a_1, m_2(a_2,a_3), a_4) =$



Così la condizione nella definizione si scrive:

$\sum_{n=r+s+t} (-1)^{r+st}$



$= 0$

RMK $\boxed{n=1}$ $\quad m_1 : A \to A$ di grado 1

$$0 = \sum_{r+s+t=1} (-1)^{r+s\cdot t} m_{r+1+t} (id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

$$= \sum_{\substack{r=0\\ s=1\\ t=0}} (-1)^{0+1\cdot 0} m_{0+1+0} (id^{\otimes 0} \otimes m_1 \otimes id^{\otimes 0})$$

 $=0$

$$= m_1(m_1)$$

$\rightsquigarrow \; m_1^2 = 0$

$\rightsquigarrow \; (A, m_1)$ complesso

$\boxed{n=2}$ $\quad m_2 : A \otimes A \to A$ di grado 0

$$0 = \sum_{r+s+t=2} (-1)^{r+s\cdot t} m_{r+1+t} (id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

$$= \sum_{\substack{(r,s,t)\in\{(1,1,0),\\ (0,2,0),\\ (0,1,1)\}}} (-1)^{r+s\cdot t} m_{r+1+t} (id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

$$= - m_2(id \otimes m_1) + m_1(m_2) - m_2(m_1 \otimes id)$$

$\rightsquigarrow \; m_2$ derivaz. di (A, m_1)

$\rightsquigarrow \; (A, m_1, m_2)$ dga

$\boxed{n=3}$ $\quad m_3 : A \otimes A \otimes A \to A$ di grado -1

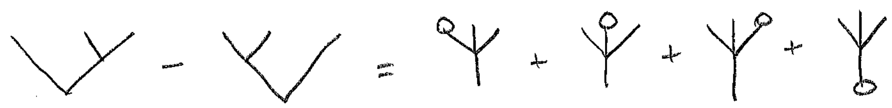
$$0 = \sum_{r+s+t=3} (-1)^{r+s\cdot t} m_{r+1+t} (id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

$$= \sum_{\substack{(r,s,t)\in\{(2,1,0),(1,2,0),\\ (1,1,1),(0,3,0),\\ (0,2,1),(0,1,2)\}}} (-1)^{r+s\cdot t} m_{r+1+t} (id^{\otimes r} \otimes m_s \otimes id^{\otimes t})$$

$$= m_3(id \otimes id \otimes m_1) - m_2(id \otimes m_2) + m_3(id \otimes m_1 \otimes id)$$
$$+ m_1(m_3) + m_2(m_2 \otimes id) + m_3(m_1 \otimes id \otimes id)$$

$\rightsquigarrow \; m_2(id \otimes m_2) - m_2(m_2 \otimes id)$

$$= m_1(m_3) + m_3(id \otimes id \otimes m_1 + id \otimes m_1 \otimes id + m_1 \otimes id \otimes id)$$



$\rightsquigarrow \; m_2$ è un prodotto associativo a meno di omotopia

RMK
- $A^\bullet$ A_∞-algebra $\rightsquigarrow$ $H^\bullet(A^\bullet, m_1)$ algebra associativa graduata
- $A^\bullet = A^0$ A_∞-algebra $\rightsquigarrow$ $m_n = 0 \;\forall n \neq 2$ $\rightsquigarrow$ $A^\bullet$ algebra associativa
- $m_n = 0 \;\forall n \geq 2 \iff (A, m_1)$ complesso
  $m_n = 0 \;\forall n \geq 3 \iff (A, m_1, m_2)$ dga

CIRM

Morfismi

DEF. Un morfismo $f: A \to B$ di A_∞-algebre è:

- $\forall n \geq 1$, $f_n: A^{\otimes n} \to B$ di grado $1-n$

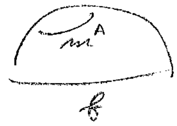
t.c.
$$\boxed{\sum_{n=r+s+t} (-1)^{r+s\cdot t} f_{r+1+t}(id^{\otimes r} \otimes m_s^A \otimes id^{\otimes t}) = \sum_{r=1}^{n} \sum_{n=i_1+\cdots+i_r} (-1)^{\sum_{l=1}^{r}(r-l)(i_l-1)} m_r^B(f_{i_1}\otimes\cdots\otimes f_{i_r}).}$$

- La composiz. di $g: A \to B$ ed $f: B \to C$ è definita da:

$$(f\circ g)_n := \sum_{r=1}^{n} \sum_{n=i_1+\cdots+i_r} (-1)^{\sum_{l=1}^{r}(r-l)(i_l-1)} f_r \circ (g_{i_1}\otimes\cdots\otimes g_{i_r}).$$

- L'identità è $id_A: A \to A$ tc $(id_A)_1 = id_A$ e $(id_A)_j = 0$.

RMK Nella rappresentazione grafica di Kontsevich:

$\sum \pm$ 

$= \sum \pm$ 

RMK $\boxed{n=1}$ $f_1: A \to B$ di grado 1

$(r,s,t) = (0,1,0)$, $(r,(i_1,\dots,i_r)) = (1,(1))$:

$$f_1(m_1^A) = m_1^B(f_1)$$

$\rightsquigarrow$ $f_1: (A, m_1^A) \to (B, m_1^B)$ morfismo di complessi

$\boxed{n=2}$ $f_2: A \to B$ di grado 0:

$(r,s,t) \in \{(1,1,0), (0,2,0), (0,1,1)\}$,

$(r,(i_1,\dots,i_r)) \in \{(1,(2)), (2,(1,1))\}$:

$$-f_2(id\otimes m_1) + f_1(m_2) - f_2(m_1\otimes id) = m_1(f_2) + m_2(f_1\otimes f_1)$$

$\rightsquigarrow$ $f_1(m_2) = m_2(f_1\otimes f_1) + m_1 f_2 + f_2(m_1\otimes id + id\otimes m_1)$

cioè f_1 commuta con la moltiplicazione m_2 a meno di un'omotopia data da f_2.

DEF. $f: A \to B$ morfismo di A_∞-algebre si dice

- quasi-isomorfismo se $f_1 : (A, m_1) \to (B, m_1)$ è qis
- stretto se $f_j = 0 \ \forall j \neq 1$.

DEF.
- Una A_∞-algebra si dice minimale se $m_1 = 0$.
- Un modello di una A_∞-algebra A è
$$f : B \xrightarrow[\text{qis}]{} A \quad \text{di } A_\infty\text{-algebre.}$$

Se $f: B \xrightarrow[\text{qis}]{} A$ è un modello minimale (cioè, $m_1^B = 0$), allora $B \simeq H(A)$.

DEF.
- Una A_∞-algebra si dice formale se ha un modello minimale che ha struttura di algebra graduata associativa (cioè, $m_n^B = 0 \ \forall n \neq 2$).
- Una A_∞-algebra si dice intrinsecamente formale se ogni A_∞-algebra con coomologia isomorfa è formale.

BAR-CONSTRUCTION:

A_∞-alg come co-alg differenziali

Co-algebre:

DEF. Un'**algebra** è
- A k-modulo
- $m : A\otimes A \to A$ moltiplicaz. associativa:

$$\begin{tikzcd} & A\otimes A\otimes A \arrow[dl, "m\otimes id"'] \arrow[dr, "id\otimes m"] & \\ A\otimes A \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

DEF. Una **derivazione** di un'algebra è $L : A \to A$ che soddisfi Leibniz:

$$\begin{tikzcd} A\otimes A \arrow[r, "L\otimes id + id\otimes L"] \arrow[d, "m"'] & A\otimes A \arrow[d, "m"] \\ A \arrow[r, "L"'] & A \end{tikzcd}$$

DEF. Una **dga** è:
- un'alg graduata (i.e., moltiplicaz. di grado 0)
- con una derivazione $d : A \to A$ di grado 1 tc $d^2=0$

DEF. Una **co-algebra** è
- C k-modulo
- $\Delta : C \to C\otimes C$ co-moltiplicaz. co-associativa:

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C\otimes C \arrow[dr, "\Delta\otimes id"'] & & C\otimes C \arrow[dl, "id\otimes\Delta"] \\ & C\otimes C\otimes C & \end{tikzcd}$$

DEF. Una **co-derivaz.** di una co-alg è $L : C \to C$ che soddisfi co-Leibniz:

$$\begin{tikzcd} C \arrow[r, "L"] \arrow[d, "\Delta"'] & C \arrow[d, "\Delta"] \\ C\otimes C \arrow[r, "L\otimes id + id\otimes L"'] & C \end{tikzcd}$$

DEF. Una **dgca** è
- una co-alg graduata (i.e., co-moltipl. di grado 0)
- con una co-derivazione $b : A \to A$ di grado 1 tc $b^2=0$.

Ex. Sia V uno sp. vett. graduato.
Def.: $\bar{T}V := \sum_{n=1}^{\infty} V^{\otimes n}$ (alg. tensoriale ridotta)

$\Delta : \bar{T}V \to \bar{T}V$, $\quad \Delta(v_1\otimes\cdots\otimes v_n) := \sum_{i=1}^{n} (v_1\otimes\cdots\otimes v_i)\otimes(v_{i+1}\otimes\cdots\otimes v_n)$.

$\rightsquigarrow$ $(\bar{T}V, \Delta)$ è una co-alg graduata.

THM (universalità). $\forall C$ co-alg graduata, $\forall C \to V$ lineare,
$\exists!$ $C \to \bar{T}V$ morf. di co-alg grad tc

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bar{T}V \arrow[d, "pr"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

LEMMA Sia V sp. vett. graduato. Si ha un isomorfismo:

$$\varphi: \mathrm{Coder}(\overline{T}V) \xrightarrow{\simeq} \mathrm{Hom}(\overline{T}V; V)$$

dove:

$$\varphi: \mathrm{Coder}(\overline{T}V) \to \mathrm{Hom}(\overline{T}V; V),$$

$$(L: \overline{T}V \to \overline{T}V) \longmapsto (\overline{T}V \xrightarrow{L} \overline{T}V \xrightarrow{\pi} V)$$

$$\varphi^{-1}: \mathrm{Hom}(\overline{T}V; V) \longrightarrow \mathrm{Coder}(\overline{T}V)$$

$$L = \{L_s: V^{\otimes s} \to V\}_{s \geq 1} \longmapsto \Big\{ \sum_{n = r+s+t} id^{\otimes r} \otimes L^s \otimes id^{\otimes t} \Big\}_{n \in \mathbb{N}}$$

DEF. A sp. vett.$/_{\mathbb{K}}$ graduato.

La sospensione $sA := A[-1]$ di A è lo sp. vett$/_{\mathbb{K}}$ graduato

$$(sA)^n := A^{n+1}.$$

c'è una mappa naturale di grado -1

$$s: A \longrightarrow sA.$$

THM La mappa

$$\{A_\infty\text{-alg}\} \longrightarrow \{dgca\}$$

$$(A, \{m_n\}_{n \geq 1}) \longmapsto \left(\overline{T}sA,\ \varphi^{-1}\left(\{s \circ m_n \circ (s^{\otimes n})^{-1}\}_{n \geq 1}\right)\right)$$

induce un'equivalenza di categorie.

HOMOTOPY TRANSFER PRINCIPLE

la costruzione di Merkulov

THM (KADEISHVILI). Sia $(A, \{m_n^A\}_n)$ una A_∞-algebra. La coomologia $H^\bullet(A, m_1)$ è dotata di una struttura $\{m_n\}_{n \geq 1}$ di A_∞-alg t.c.:

- $m_1 = 0$
- m_2 è indotto da m_2^A
- esiste $H^\bullet(A) \xrightarrow[\text{qis}]{} A$ qis di A_∞-algebre che induce l'identità $id_{H^\bullet(A)}$ in coomologia.

Tale struttura è unica a meno di (non-unici) isom. di A_∞-alg.

Costruzione: Scegliamo un retratto di deformazione:

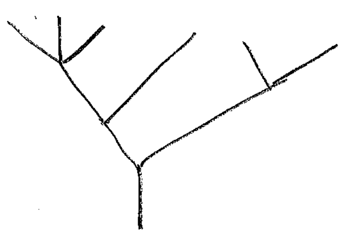
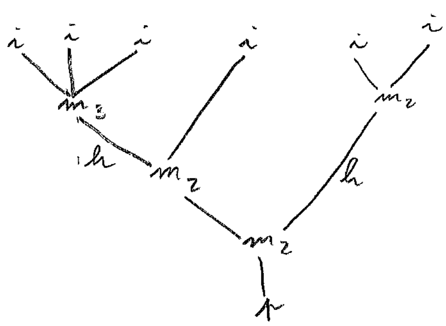
$$h[-1] \circlearrowright A \underset{i[0]}{\overset{p[0]}{\rightleftarrows}} H(A)$$

i.e.: $p\, i = id_{H(A)}$

$i\, p - id_A = [m_1^A, h]$

Def. $$m_n^{H(A)} := \sum_{\substack{\text{rooted planar graphs} \\ \text{con } n \text{ foglie}}} (\{m_n^A\})^T$$

dove $(\{m_n^A\})^T$ è ottenuto dalla composizione del grafo T scrivendo: i su ogni foglia;
m_n su ogni nodo con n branche;
h su ogni lato interno
p sulla radice.

ex. $T =$  $\rightsquigarrow$ 

$$\rightsquigarrow (\{m_n^A\})^T = p\, m_2(\, m_2(\, h\, m_3(i \otimes i \otimes i) \otimes i\,) \otimes h\, m_2(i \otimes i))$$

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Merkulov model:

Vediamo la costruz. esplicita nel caso di una dga.

ref.
- MERKULOV, Strong homotopy algebras of a Kähler manifold IMRN 3 (1999), 153-164
- ZHOU, Hodge theory and A_∞ structures on cohomology, IMRN
- LU, PALMIERI, WU, ZHANG, A-infinity structures on Ext-algebras, J. PURE APPL. ALGEBRA 213 (2009), 2017-2037

Sia: $(A, d, \cdot)$ dga.

Consideriamo: $B^n = \mathrm{im}\, d \cap A^n$, $Z^n = \ker d \cap A^n$.

$\exists\ H^n, L^n$ tc
$$Z^n = B^n \oplus H^n$$
$$A^n = Z^n \oplus L^n = B^n \oplus H^n \oplus L^n$$

Denotiamo:
$$i : H^\circ(A) \xrightarrow[\text{qis}]{[0]} A \quad \text{tc} \quad \mathrm{im}(i) = H$$
$$p : A \longrightarrow H = H(A)$$

Scegliamo: $G : A \xrightarrow{[-1]} A$ omotopia tra id_A ed ip:
$$id_A - ip = [d, G]$$
in modo tale che:
$$G^n|_{L^n \oplus H^n} = 0$$
$$G^n|_{B^n} = (d|_{L^{n-1}})^{-1}$$

Def.: $\lambda_n : A^{\otimes n} \longrightarrow A$ mappa lineare di grado $2-n$:
$$\boxed{\begin{cases} G\lambda_1 = -id_A \\ \lambda_2(a_1 \otimes a_2) = a_1 \cdot a_2 \\ \lambda_n = \sum\limits_{n=s+t} (-1)^{s+1} \lambda_2(G\lambda_s \otimes G\lambda_t) \quad \text{per } n \geq 3 \end{cases}}$$

Def.: $\boxed{m_n := p\lambda_n}$

THM (MERKULOV). $(H(A), \{m_n\}_n)$ A_∞-algebra.

Def.: $f_n : -G\lambda_n : (H(A))^{\otimes n} \longrightarrow A$

PROP. $\{f_n\}_n : (H(A), \{m_n\}_n) \xrightarrow[\text{qis}]{} (A, d, \cdot)$

CIRM

DEF. $(H(A), \{m_n\}_n)$ A_∞-alg costruita come sopra è detta <u>modello di Merkulov</u> della dga $(A, d, \cdot)$.

THM (LU, PALMIERI, WU, ZHANG). Ogni A_∞-algebra minimale (i.e., $m_1 = 0$) è modello di Merkulov di una dga.

Massey products:

$(A, d, \cdot)$ dga.

NOT. $\bar{a} := (-1)^{1+\deg a} a$

DEF. • length-3 Massey products:

siano $\alpha_1, \alpha_2, \alpha_3 \in HA$:

$\rightsquigarrow \alpha_1 = [a_{01}]$, $\alpha_2 = [a_{12}]$, $\alpha_3 = [a_{23}]$

supp. $\alpha_1\alpha_2 = 0$, $\alpha_2\alpha_3 = 0$

$\rightsquigarrow \overline{a_{01}} a_{12} = d a_{02}$, $\overline{a_{12}} a_{23} = d a_{13}$

def. : $\overline{a_{02}} a_{23} + \overline{a_{01}} a_{13} \in \ker d$ Massey triple product

def. $a_{0123} =: \langle \alpha_1, \alpha_2, \alpha_3 \rangle := \{ [\overline{a_{02}} a_{23} + \overline{a_{01}} a_{13}] \in H(A)$: al variare di $a_{01}, a_{12}, a_{23} \}$

$$\in H(A) \big/ \langle \alpha_1, \alpha_3 \rangle$$

• length-n Massey products:

siano $\alpha_1, \dots, \alpha_n \in H(A)$

$\rightsquigarrow \alpha_j = [a_{j-1,j}]$

supp.: $\forall\, i < j$ tc $j-1 < n-1$, ogni length-$(j-i+1)$ Massey products è definito e contiene lo 0:

$$\rightsquigarrow d\, a_{ij} = \sum_{i<k<j} \overline{a_{ik}}\, a_{kj}$$

def. n-length Massey product

$$\langle \alpha_1, \text{—}, \alpha_n \rangle := \Big\{ \sum_{0<i<n} \overline{a_{0i}}\, a_{in} \Big\}.$$

THM (LU, PALMIERI, WU, ZHANG).

Sia $(A, d, \cdot)$ dga. A meno del segno, la struttura A_∞ indotta su HA dà i prodotti di Massey:

$$(-1)^{1+\deg\alpha_{n-1}+\deg\alpha_{n-3}+\cdots}\, m_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

$$\in \langle \alpha_1, \text{—}, \alpha_n \rangle.$$

Esempio: KODAIRA – THURSTON algebra

CIRM

Kodaira – Thurston:

$A = \Lambda^\bullet \mathbb{C}\langle a, b, c, x\rangle$ dove: $\deg a = \deg b = \deg c = \deg x = 1$,
$da = db = dc = 0$, $dx = a \cdot b$.

$$A = \underbrace{\langle 1, a, b, c, ac, ax, bc, bx, abx, acx, bcx, abcx\rangle}_{H = H^\bullet(A)} \oplus \underbrace{\langle x, cx\rangle}_{L} \oplus \underbrace{\langle ab, abc\rangle}_{B}$$

$G[-1] \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H(A)$:

$*$	$p(*)$	$G(*)$	$*$	$p(*)$	$G(*)$
1	1	0	$abcx$	$abcx$	0
a	a	0	abc	0	cx
b	b	0	abx	abx	0
c	c	0	acx	acx	0
x	0	0	bcx	bcx	0
ab	0	x	bc	bc	0
ac	ac	0	bx	bx	0
ax	ax	0	cx	0	0

$G\lambda_1 := -id_A$

$\lambda_2(\alpha\otimes\beta) := \alpha\cdot\beta$

$$\lambda_3 := \sum_{3=s+t} (-1)^{s+1} \lambda_2(G\lambda_s \otimes G\lambda_t)$$

$$= -\lambda_2(G\lambda_2 \otimes G\lambda_1) + \lambda_2(G\lambda_1 \otimes G\lambda_2)$$

$$= \lambda_2(G\lambda_2 \otimes id) - \lambda_2(id \otimes G\lambda_2)$$

$m_3 := p\lambda_3$

$$\rightsquigarrow\ m_3(\alpha\otimes\beta\otimes\gamma) = p\lambda_2(G\lambda_2(\alpha\otimes\beta)\otimes\gamma) - p\lambda_2(\alpha\otimes G\lambda_2(\beta\otimes\gamma))$$

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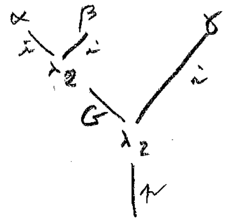

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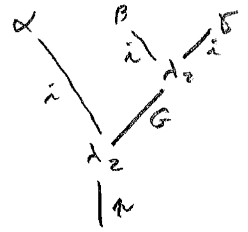

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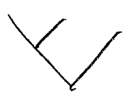

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$$\operatorname{imm} m_3 = \operatorname{imm}\left[p\lambda_2(G\lambda_2\otimes \mathrm{id}) - p\lambda_2(\mathrm{id}\otimes G\lambda_2)\right]$$

$$\underset{\substack{\operatorname{imm} G\lambda_2 \\ = \operatorname{imm} G \\ = \langle x, cx\rangle}}{\subseteq} \operatorname{imm}\left[p\lambda_2(\langle x, cx\rangle \otimes \mathrm{id}) - p\lambda_2(\mathrm{id}\otimes \langle x, cx\rangle)\right]$$

$$\subseteq \operatorname{imm}\left[p\langle \cancel{x}, ax, bx, \cancel{cx}, abx, acx, bcx, abcx\rangle\right]$$

$$= \langle ax, bx, abx, acx, bcx, abcx\rangle$$

Osserviamo che:

$$\operatorname{imm} G\lambda_3 \subseteq \operatorname{imm} G\langle x, ax, bx, cx, abx, acx, bcx, abcx\rangle = \{0\}.$$

$$\lambda_4 := \sum_{4=r+s} (-1)^{r+1}\lambda_2(G\lambda_r\otimes G\lambda_s)$$

$$= \lambda_2(G\lambda_3\otimes G\lambda_1) - \lambda_2(G\lambda_2\otimes G\lambda_2) + \lambda_2(G\lambda_1\otimes G\lambda_3)$$

$$= -\lambda_2(G\lambda_3\otimes \mathrm{id}) - \lambda_2(G\lambda_2\otimes G\lambda_2) - \lambda_2(\mathrm{id}\otimes G\lambda_2)$$

$$= -\lambda_2(G\lambda_2(G\lambda_2\otimes \mathrm{id})\otimes \mathrm{id}) + \lambda_2(G\lambda_2(\mathrm{id}\otimes G\lambda_2)\otimes \mathrm{id})$$

$$-\lambda_2(G\lambda_2\otimes G\lambda_2)$$

$$-\lambda_2(\mathrm{id}\otimes G\lambda_2(G\lambda_2\otimes \mathrm{id})) + \lambda_2(\mathrm{id}\otimes G\lambda_2(\mathrm{id}\otimes G\lambda_2))$$

$m_4 := p\lambda_4 = -$ 

$$\operatorname{imm} m_4 = \operatorname{imm} p\lambda_4 \subseteq \operatorname{im} p\lambda_2(G\lambda_2(G\lambda_2\otimes \mathrm{id})\otimes \mathrm{id}) + \operatorname{im} p\lambda_2(G\lambda_2\otimes G\lambda_2)$$

$$\subseteq \operatorname{im} p\lambda_2(G\langle x, ax, bx, cx, abx, acx, bcx, abcx\rangle) + \operatorname{im} p\lambda_2(\langle x, cx\rangle\otimes\langle x, cx\rangle)$$

$$= \{0\}$$

$$\rightsquigarrow \quad \lambda_4 = m_4 = 0.$$

CIRM

Per $n \geq 3$:

$$\lambda_n := \sum_{n=r+k} (-1)^{r+1} \lambda_2 (G\lambda_r \otimes G\lambda_k)$$

$$= \sum_{\substack{n=r+k \\ r \leq 2 \\ k \leq 2}} (-1)^{r+1} \lambda_2 (G\lambda_r \otimes G\lambda_k)$$

$$= 0$$

Riassumendo:

$$m_2(\alpha \otimes \beta) = p(\alpha \cdot \beta)$$

$$m_3 = p\,\lambda_2 (G\lambda_2 \otimes id) - p\,\lambda_2 (id \otimes G\lambda_2)$$

$$m_n = 0 \qquad \forall n \geq 3.$$

<u>Quindi</u>: i length-n Massey prod su K-T sono nulli $\forall\, n \geq 4$.

$\exists$ length-3 Massey prod non nulli sono:

$$\langle [a],[a],[b] \rangle = [ax] + \langle [a],[ac],[abx],[abcx],[b],[bc] \rangle$$
$$\langle [b],[a],[b] \rangle = [2bx] + \langle [b],[bc],[abx],[abcx] \rangle$$
$$\langle [ac],[a],[b] \rangle = [-acx] + \langle [ac],[abcx],[b],[bc],[abx] \rangle$$
$$\langle [a],[b],[a] \rangle = [-2ax] + \langle [a],[ac],[abx],[abcx] \rangle$$
$$\langle [a],[b],[b] \rangle = [-xb] + \langle [a],[ac],[abx],[abcx],[b],[bc] \rangle$$
$$\langle [a],[b],[ac] \rangle = [2acx] + \langle [a],[ac],[abx],[abcx] \rangle$$
$$\langle [a],[b],[bc] \rangle = [bcx] + \langle [a],[ac],[abx],[abcx],[bc] \rangle$$
$$\langle [a],[a],[bc] \rangle = [acx] + \langle [a],[ac],[abx],[abcx],[bc] \rangle$$
$$\langle [a],[bc],[a] \rangle = [-2acx] + \langle [a],[ac],[abx],[abcx] \rangle$$
$$\langle [a],[bc],[b] \rangle = [-bcx] + \langle [a],[ac],[abx],[abcx],[b],[bc] \rangle$$
$$\langle [a],[b],[ac] \rangle = [acx] + \langle [a],[ac],[abx],[abcx] \rangle$$
$$\langle [b],[b],[ac] \rangle = [bcx] + \langle [b],[bc],[abx],[abcx],[ac] \rangle$$
$$\langle [b],[ac],[a] \rangle = [-acx] + \langle [a],[ac],[abx],[abcx],[b],[bc] \rangle$$
$$\langle [b],[ac],[b] \rangle = [-2bcx] + \langle [b],[bc],[abx],[abcx] \rangle$$

<u>Ricordiamo</u>:

$\bar{a} \cdot b = d(-x)$	$\bar{b}\, a = d(x)$
$\bar{b} \cdot ac = d(-cx)$	$\overline{ac} \cdot b = d(cx)$
$\bar{a} \cdot bc = d(-cx)$	$\overline{bc} \cdot a = d(cx)$

$$H^\bullet(A) = \langle 1 \rangle \oplus \langle a, b, c \rangle \oplus \langle ac, ax, bc, bx \rangle \oplus \langle abx, acx, bcx \rangle \oplus \langle abcx \rangle$$

<u>RMK</u> In [LU-PALMIERI-WU-ZHANG; Example 6.5], un esempio simile è studiato per mostrare come, per ogni elemento di un prodotto di Massey, si può scegliere una struttura A_∞ derivante dalla costruzione di Merkulov t.c. l'elemento sia proprio un prodotto.

(<u>QUEST</u> È un fatto generale?)

<u>PROBLEMA</u> Caratterizzare la formalità in termini di una GAUGE HODGE CONDITION, come in:

DOTSENKO, SHADRIN, VALLETTE, De Rham cohomology and homotopy Frobenius manifolds, arXiv: 1203.5077 [math.KT].

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